MATH H53 : PRACTICE PROBLEMS FOR MID-TERM-2

- Chapter 14 Review (pg. 968) - 9, 31, 47, 51, 53, 55, 56, 63
- Section 15.10 - 17, 19, 23, 25, 28
- Chapter 15 Review (pg. 1050) - 13, 21, 27, 31, 33, 36, 42, 50

Additional problems :

(1) Consider the function defined by

$$f(x,y) = \begin{cases} \frac{\sin(xy)}{x}, & x \neq 0 \\ y, & x = 0 \end{cases}$$

(a) State the $\epsilon - \delta$ definition of continuity.
(b) Use the above definition to show that the function is continuous everywhere. **Hint:** At points where $x = 0$, you can use Taylor's inequality $|\sin\theta - \theta| \leq \frac{\theta^2}{2}$.
(c) What is $\nabla f(0,0)$?
(d) Find the directional derivatives $D_{\mathbf{u}} f(0,0)$. **Note:** If you want to use Chain rule, you should first prove that the function is differentiable. It might just be easier to calculate the directional derivative by the limit definition.

(2) Show that the function defined by

$$f(x,y) = \begin{cases} \frac{\sin(xy^2)}{x^2+y^4}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

is not differentiable. **Hint:** Is it continuous?

(3) Let $f(x,y)$ be continuous in an open set containing the point (a,b). If $f(a,b) > 0$, show that there exists a small disc around (a,b) such that the function is always positive in this small disc.

(4) What volume of material is removed from a solid sphere of radius $2r$ if a cylindrical hole of radius r is drilled through the center.

(5) Find the absolute max and min of the function $f(x,y) = x^3 + y^3 - 3xy + 1$ on the disc $x^2 + y^2 \leq 1$.

(6) Show that the volume of the solid bounded above by $r^2 + z^2 = a^2$ and below by cone $z = r\cot\varphi_0$, where $r^2 = x^2 + y^2$ and $0 < \varphi_0 < \pi/2$, is given by

$$V = \frac{2\pi a^3}{3}(1 - \cos\varphi_0).$$

Note that as the cone opens up (i.e. $\varphi_0 \to \pi/2$), V tends towards the volume of the hemisphere.