MATH H53 : PRACTICE PROBLEMS FOR MID-TERM-1

- Section 13.1 - 29, 43, 47
- Section 13.2 - 27, 55,
- Section 13.3 - 17, 24, 42, 49 (only osculating plane), 51
- Section 13.4 - 39
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Additional problems :

(1) (a) If $\mathbf{r}'(t) \neq 0$, then show that

$$\frac{d}{dt}|\mathbf{r}(t)| = \frac{1}{|\mathbf{r}(t)|}\mathbf{r}(t) \cdot \mathbf{r}'(t).$$

(b) Using this, show that if a particle moves with such that its position vector is always perpendicular to it's velocity vector, then the particle moves on a sphere centered at the origin.

(2) Suppose a planet of mass m revolves around a sun of mass M (which we think of lying at the origin). The by Newton's law of gravitation, the acceleration is given by

$$\mathbf{a} = -\frac{GM}{|\mathbf{r}|^3}\mathbf{r},$$

and is in particular radial.

(a) Show that $\mathbf{r}$ and $\mathbf{r} \times \mathbf{r}'$ are orthogonal. Hence show that the trajectory of the planet lies in a plane containing the sun.

Note: If one could also show that the trajectory lies on a cone, this would imply that the trajectory is a conic section. Trajectories of the planets are bounded, and the only bounded sections are ellipses, and hence this would prove Kepler's first law. But I am unable to directly see why the trajectory would be a cone. This is probably the reason that the usual proofs are quite different from this!

(b) Show that the energy and angular momentum, defined respectively as

$$E = \frac{1}{2}m|\mathbf{r}'|^2 - \frac{GMm}{|\mathbf{r}|}, \ \mathbf{L} = m\mathbf{r} \times \mathbf{r}',$$

are conserved i.e $\frac{dE}{dt} = 0$ and $\frac{d\mathbf{L}}{dt} = \mathbf{0}$.

(3) Show that the triangle with vertices $P(2,3,0)$, $Q(1,2,1)$ and $R(2,1,1)$ form a right angled triangle.

(4) Find the equation of a plane passing through $(1,1,1)$ and whose normal makes angles $\frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{3}$ with the basis vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$.