Last lecture, we define the complex exponential function by the power series
\[ e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}. \]
We saw that the power series has infinite radius of convergence, and hence defines a function on the entire complex plan. In fact by the theorem from last lecture, we now know that the exponential function is holomorphic on the entire plain, and hence an entire function. Moreover, to find the complex derivative, it is enough to differentiate term-wise. So
\[ \frac{d}{dz} e^z = \sum_{n=0}^{\infty} \frac{1}{n!} 
\frac{d}{dz} z^n = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{z^n}{n!}. \]
To see the last equality just replace \( n - 1 \) by \( n \) in the penultimate term. So we see that
\[ \frac{d}{dz} e^z = e^z. \]
In fact this property characterizes the exponential function, and we have the following theorem.

**Theorem 0.1.** There is a unique holomorphic function \( f : \mathbb{C} \to \mathbb{C} \) satisfying
\[ \begin{cases} 
    f'(z) = f(z) \\
    f(0) = 1.
\end{cases} \]
We defer the proof to the end of the lecture. We now collect some basic properties of the exponential function.

**Theorem 0.2.** The exponential satisfies the following properties.

1. For any complex numbers \( z, w \) we have that
   \[ e^{z+w} = e^z e^w. \]
2. \( (\text{Euler’s identity}) \) For any real number \( x \),
   \[ e^{ix} = \cos x + i \sin x. \]
(3) **(Periodicity)** For any $z \in \mathbb{C}$, and $n \in \mathbb{Z}$,

$$e^{z+2\pi in} = e^z,$$

and so $e^z$ is periodic with a period of $2\pi i$.

(4) $e^z = 1$ if and only if $z = 2n\pi i$ for some integer $n$.

(5) $e^z \neq 0$ for all $z \in \mathbb{C}$.

**Proof.** (1) This follows from the product formula for power series and the binomial theorem. The left hand side of the equation is

$$e^{z+w} = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!}.
$$

But then

$$\frac{(z+w)^n}{n!} = \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} z^k w^{n-k} = \sum_{k=0}^{n} \frac{z^k}{k!} \frac{w^{n-k}}{(n-k)!}.
$$

This is exactly the $n^{th}$ coefficient of the Cauchy product, and the result follows.

(2) To see this, note that

$$i^n = \begin{cases} (-1)^m, & n = 2m \\ (-1)^m i, & n = 2m + 1. \end{cases}
$$

So then by definition

$$e^{ix} = \sum_{n=0}^{\infty} \frac{i^n x^n}{n!} = \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m}}{(2m)!} + i \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+1}}{(2m+1)!}.
$$

Euler’s identity then follows from the observation that the two series on the right are simply the Maclaurin series for sine and cosine respectively.

(3) This follows trivially from the Euler identity, and the periodicity of the cosine and sine functions.

(4) For any $z = x + iy$, by the first two properties

$$e^z = e^x (\cos y + i \sin y).
$$

So then $e^z = 1$ if and only if $e^x = 1$, $\sin y = 0$, $\cos y = 1$. That is if and only if $x = 0$ and $y = 2n\pi i$.

(5) Like above, $e^z = 0$ if and only if $e^x = 0$ or $\cos y + i \sin y = 0$. But neither can ever be zero! \[\square\]
Remark 1. Polar coordinates. The Euler identity can be used to give a third representation of complex numbers in terms of the exponential function. Namely, for \( z \in \mathbb{C} \), let \( r = |z| \) and \( \theta = \arg z \). Then we have seen that
\[
z = r \cos \theta + ir \sin \theta.
\]
So by the Euler identity, we have the representation
\[
z = re^{i\theta}.
\]
This is sometimes very useful in computations.

0.1. Trigonometric functions. We can now analogously define the functions sine and cosine using power series:
\[
\cos z = \sum_{m=0}^{\infty} (-1)^m \frac{z^{2m}}{(2m)!},
\]
\[
\sin z = \sum_{m=0}^{\infty} (-1)^m \frac{z^{2m+1}}{(2m+1)!}.
\]
It is easy to check that the radius of convergence of both the power series is infinity, and hence they define entire functions, just like the exponential function. In fact an easy computation also shows that
\[
\frac{d}{dz} \cos z = -\sin z, \quad \frac{d}{dz} \sin z = \cos z.
\]
The same computation then gives the following generalized Euler identity.

Proposition 0.1 (Generalized Euler identity). For any \( z \in \mathbb{C} \),
\[
e^{iz} = \cos z + i \sin z.
\]

Next, we collect some properties of the sine and cosine functions. These can be proved using the generalized Euler identity, and the analogous properties of the exponential function.

Theorem 0.3. The sine and cosine function satisfy the following.

1. For \( z, w \in \mathbb{C} \),
\[
\sin (z \pm w) = \sin z \cos w \pm \cos z \sin w,
\]
\[
\cos (z \pm w) = \cos z \cos w \mp \sin z \sin w.
\]

2. Sine and Cosine are periodic with periods \( 2\pi \), i.e.
\[
\sin (z + 2\pi) = \sin z, \quad \cos (z + 2\pi) = \cos z.
\]

One can also define Tangent and Cotangent functions in the usual way.
For functions of one real variable, the logarithm is the inverse function of the exponential function. We would like to generalize this to complex numbers. In particular, we would like to have a definition for logarithm that makes it a holomorphic function. An immediate difficulty is that while on real line, the exponential function is strictly increasing, and hence one-one, on the complex plane, we have already seen that the exponential function is not one-one. For instance \( e^0 = e^{2\pi i n} = 1 \) for all \( n \in \mathbb{Z} \). So to define an inverse function, one has to make a choice of the pre-image. For instance, we can choose to log 1 = 0 or \( 2\pi i n \).

In fact, writing in polar coordinates \( z = re^{i\theta} \), \( f(z) \) satisfies \( e^{f(z)} = f(e^z) = z \) on any open connected set if and only if

\[
f(z) = \log r + i\theta + 2\pi in, \quad n = 0, 1, -1, 2, -2, \ldots.
\]

That is, we can at best define logarithm as multivalued function. A choice of \( n \) corresponds to defining a single valued logarithm, the corresponding function is called a branch of the logarithm. For instance, choosing \( n = 0 \), and defining \( \log z = \log r + i\theta \), picks out what is called as the principal branch of the logarithm. This might seem like a good definition until we realize that the logarithm so defined is not even continuous. To see this, suppose \( z \to -1 \) from the 2nd quadrant. Then \( \log z \) will tend to \( i\pi \). But on the other hand, as \( z \to -1 \) from the third quadrant, \( \log z \) will tend to \( -\pi \). This is not only a minor irritant that can be fixed by some trick, but as we will see later in the course, is a fundamental issue. In fact, we will see that there is actually no way to define a holomorphic logarithm on which is defined on all of \( \mathbb{C} \setminus \{0\} \). The best we can do is to define it outside of a ray. In fact we have the following.

**Proposition 0.2.** The function

\[
\log z := \log |z| + i\arg z
\]

defines a holomorphic function on \( \mathbb{C} \setminus \Re(z) \leq 0 \) satisfying \( e^{\log z} = z \). Moreover, applying chain rule, we see that

\[
\frac{d}{dz} \log z = \frac{1}{z}.
\]

We will see a proof of this later in the course. The logarithm function then has the property that \( \log 1 = 0 \) and

\[
\log zw = \log z + \log w,
\]

assuming it is defined at all those points.
Logarithm as power series. Since $e^0 = 1$, and definition of logarithm must satisfy $\log 1 = 0$. Let’s see if we can have a definition of logarithm using power series centered at $z_0 = 1$. By the chain rule, if have a holomorphic function $\log z$ near $z_0 = 1$, then

$$\frac{d}{dz} \log z = \frac{1}{z}.$$ 

Iteratively, we must have

$$\frac{d^n}{dz^n} \log z = (-1)^{n-1} (n-1)!.$$ 

If $\log z$ has a power series expansion around $z_0 = 1$, then the coefficients must be given by

$$a_n = \frac{1}{n!} \frac{d^n}{dz^n} \log z \bigg|_{z=1} = (-1)^{n-1} \frac{1}{n}.$$ 

Turning this around, we consider the power series

$$F(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n.$$ 

We then have the following

**Theorem 0.4.** The holomorphic function $F(z): D_1(1) \to \mathbb{C}$ satisfies

$$F(z) = \log z,$$

where $\log z$ is the principal branch of the logarithm on $\mathbb{C} \setminus \text{Re}(z) \leq 0$ defined above.

The proof is in the appendix below.

**Complex powers.** It should also be remarked, that once logarithm is defined, one can also define the powers of complex to other complex numbers by the simple formula

$$z^w = e^{w \log z}.$$ 

**Appendix: Proofs of theorems 0.1 and 0.4**

**Proof of theorem 0.1.** The proof uses the fact that if a holomorphic function has complex derivative identically zero, then the function has to be a constant. We will prove this fact later in the course. Assuming this, consider the function

$$g(z) = e^{-z} f(z).$$

Then by the Chain rule, since $f'(z) = f(z)$ we see that

$$g'(z) = -e^{-z} f(z) + e^z f'(z) = -e^{-z} f(z) + e^{-z} f(z) = 0.$$ 

Hence $g(z)$ is a constant. But by the initial condition we see that $g(0) = 1$. On the other hand by the first property in Theorem 0.2, $e^{-z} = 1/e^z$, and so $f(z) = e^z$. It is easy to see that this converges in the disc $|z-1| < 1$. 

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**Proof of theorem 0.4.** We already know that \( d\log z/dz = 1/z \) for the principal branch of the logarithm. Also, \( F(1) = \log 1 = 0 \). So, similar to the above proof, all we need to show (modulo the theorem on identically zero derivatives to be covered later) is that \( F'(z) = 1/z \). Since \( F(z) \) is a power series, by term-wise differentiation,

\[
F'(z) = \sum_{n=1}^{\infty} (-1)^{n-1}(z - 1)^{n-1} = \sum_{n=0}^{\infty} (-1)^n(z - 1)^n.
\]

From the geometric series expansion, we know that for \( |w| < 1 \),

\[
\sum_{n=0}^{\infty} w^n = \frac{1}{1-w}.
\]

Putting \( w = 1 - z \) (we can do this since \( |z - 1| < 1 \)) in the above expansion

\[
\frac{1}{z} = \sum_{n=0}^{\infty} (1 - z)^n = \sum_{n=0}^{\infty} (-1)^n(z - 1)^n = F'(z).
\]

*Department of Mathematics, UC Berkeley
E-mail address: vvdatar@berkeley.edu