LECTURE-4 : LIMITS AND CONTINUITY

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Let $S \subset \mathbb{C}$ be a subset. A function $f : S \to \mathbb{C}$ is a rule that assigns *unique* complex number, denoted by $f(z)$ to every number $z \in S$. The set S is called the *domain* of the function, and

$$f(S) := \{f(z) \mid z \in S\},$$

is called the *range*. The *pre-image* of a set $T \subset \mathbb{C}$, denoted by $f^{-1}(T)$ is the subset of S defined by

$$f^{-1}(T) = \{z \in S \mid f(z) \in T\}.$$

A function is called *injective* or *one-one* if the pre-image of every point in the range consists of exactly one point, i.e

$$f(z) = f(w) \implies z = w.$$

It is said to *surjective* or *onto* if the range is all of $\mathbb{C}$.

We say that the limit of $f(z)$ as z tends towards p is L, and denote it by

$$\lim_{z \to p} f(z) = L,$$

if the following holds - For any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|z - p| < \delta,\ z \in S, \implies |f(z) - L| < \varepsilon.$$

We say that f is *continuous* at $p \in S$ if

$$\lim_{z \to p} f(z) = f(p).$$

f is simply called continuous if it is continuous at all points in its domain. We then have the basic fact.

Theorem 0.1. *$f : S \to \mathbb{C}$ is continuous if and only if $Re(f)$ and $Im(f)$ a continuous as real valued functions of two variables.*

So as far as topology, which is the study of continuous functions, is concerned, there is no difference between $\mathbb{C}$ and $\mathbb{R}^2$. With this remark, the following properties follow easily from what is already known about multi-variable functions.

Theorem 0.2. *Consider a function $f : \Omega \to \mathbb{C}$, where Ω is open.*

(1) *It is continuous if and only if $f^{-1}(U)$ is open for any open set $U \subset \mathbb{C}$.*
(2) *It is continuous if and only if $f^{-1}(K)$ is closed for every closed set $K \subset \mathbb{C}$.*

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(3) *It is continuous at $p \in \Omega$ if and only if for any sequence $\{z_n\}$ such that $z_n \to p$, we have*

$$\lim_{z_n \to p} f(z_n) = f(p).$$

(4) *If f is continuous, then for any compact subset $K \subset \Omega$, $f(K)$ is compact.*
(5) *If f and $g : \Omega \to \mathbb{C}$ are continuous at p then so are $f \pm g$ and fg. If $g(p) \neq 0$, then f/g is also continuous at p.*
(6) *if f is continuous at p, and $g : f(\Omega) \to \mathbb{C}$ is continuous at $f(p)$, then the composition $g \circ f$ is also continuous at p.*

Example 0.1. *The function $f(z) = z^n$, where n is an integer, is continuous. To see this, note that*

$$z^n - p^n = (z - p)(z^{n-1} + z^{n-2}p \cdots + p^{n-1})$$

A polynomial is a function $p : \mathbb{C} \to \mathbb{C}$ of the form

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

where $a_k \in \mathbb{C}$ for $k = 0, 1, \cdots, n$. Then by the fact that sums of contiuous functions are continuous, it follows that polynomials are continuous at all points. A rational function is a quotient of two polynomials

$$R(z) = \frac{p(z)}{q(z)},$$

wherever q is non-zero. At all such points, by the quotient rule above, a rational function is also continuous.

Example 0.2. *The function $f(z) = \bar{z}$ is continuous. Similarly, the function $g(z) = |z|$ is also continuous.*

Example 0.3. Arg(z) in not continuous. *Recall that if $z = x + iy$, then* $\arg(z)$ *is defined as the unique angle between $(-\pi, \pi]$ that the line joining the origin to (x, y) makes with the positive x-axis. Now consider any point on the the negative x-axis, say $z = -1$.*

0.1. **Convergence of functions.** There are two notions of convergence, that of point-wise, and uniform convergence. We say that

- the sequence of functions $f_n : \Omega \to \mathbb{C}$ converges *point-wise* to $f : \Omega \to \mathbb{C}$, if for every $z \in \Omega$, the sequence $f_n(z) \to f(z)$. Or equivalently, given any $\varepsilon > 0$, and any $z \in \Omega$, there exists an $N > 0$, possibly depending both on ε and z, such that

$$n > N \implies |f_n(z) - f(z)| < \varepsilon.$$

- the sequence of functions is said to converge *uniformly* if given any $\varepsilon > 0$, there exists an $N > 0$ depending only on ε such that for all $z \in \Omega$ and $n > N$ we have that

$$|f_n(z) - f(z)| < \varepsilon.$$

Theorem 0.3. *If $f_n : \Omega \to \mathbb{C}$ is a sequence of continuous functions which converge uniformly to $f : \Omega \to \mathbb{C}$, then f itself is continuous.*