LECTURE-4 : LIMITS AND CONTINUITY

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Let $S \subset \mathbb{C}$ be a subset. A function $f : S \to \mathbb{C}$ is a rule that assigns unique complex number, denoted by $f(z)$ to every number $z \in S$. The set $S$ is called the domain of the function, and

$$f(S) := \{f(z) \mid z \in S\},$$

is called the range. The pre-image of a set $T \subset \mathbb{C}$, denoted by $f^{-1}(T)$ is the subset of $S$ defined by

$$f^{-1}(T) = \{z \in S \mid f(z) \in T\}.$$

A function is called injective or one-one if the pre-image of every point in the range consists of exactly one point, i.e

$$f(z) = f(w) \implies z = w.$$

It is said to surjective or onto if the range is all of $\mathbb{C}$.

We say that the limit of $f(z)$ as $z$ tends towards $p$ is $L$, and denote it by

$$\lim_{z \to p} f(z) = L,$$

if the following holds - For any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|z - p| < \delta, \ z \in S, \implies |f(z) - L| < \varepsilon.$$

We say that $f$ is continuous at $p \in S$ if

$$\lim_{z \to p} f(z) = f(p).$$

$f$ is simply called continuous if it is continuous at all points in its domain. We then have the basic fact.

**Theorem 0.1.** $f : S \to \mathbb{C}$ is continuous if and only if $Re(f)$ and $Im(f)$ a continuous as real valued functions of two variables.

So as far as topology, which is the study of continuous functions, is concerned, there is no difference between $\mathbb{C}$ and $\mathbb{R}^2$. With this remark, the following properties follow easily from what is already known about multi-variable functions.

**Theorem 0.2.** Consider a function $f : \Omega \to \mathbb{C}$, where $\Omega$ is open.

1. It is continuous if and only if $f^{-1}(U)$ is open for any open set $U \subset \mathbb{C}$.
2. It is continuous if and only if $f^{-1}(K)$ is closed for every closed set $K \subset \mathbb{C}$.
(3) It is continuous at \( p \in \Omega \) if and only if for any sequence \( \{z_n\} \) such that \( z_n \to p \), we have
\[
\lim_{z_n \to p} f(z_n) = f(p).
\]

(4) If \( f \) is continuous, then for any compact subset \( K \subset \Omega \), \( f(K) \) is compact.

(5) If \( f \) and \( g: \Omega \to \mathbb{C} \) are continuous at \( p \) then so are \( f \pm g \) and \( fg \). If \( g(p) \neq 0 \), then \( f/g \) is also continuous at \( p \).

(6) If \( f \) is continuous at \( p \), and \( g: f(\Omega) \to \mathbb{C} \) is continuous at \( f(p) \), then the composition \( g \circ f \) is also continuous at \( p \).

Example 0.1. The function \( f(z) = z^n \), where \( n \) is an integer, is continuous. To see this, note that
\[
z^n - p^n = (z - p)(z^{n-1} + z^{n-2}p + \cdots + p^{n-1})
\]
A polynomial is a function \( p: \mathbb{C} \to \mathbb{C} \) of the form
\[
p(z) = a_nz^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0,
\]
where \( a_k \in \mathbb{C} \) for \( k = 0, 1, \ldots, n \). Then by the fact that sums of continuous functions are continuous, it follows that polynomials are continuous at all points. A rational function is a quotient of two polynomials
\[
R(z) = \frac{p(z)}{q(z)},
\]
wherever \( q \) is non-zero. At all such points, by the quotient rule above, a rational function is also continuous.

Example 0.2. The function \( f(z) = \bar{z} \) is continuous. Similarly, the function \( g(z) = |z| \) is also continuous.

Example 0.3. Arg(z) is not continuous. Recall that if \( z = x + iy \), then \( \arg(z) \) is defined as the unique angle between \((-\pi, \pi]\) that the line joining the origin to \((x, y)\) makes with the positive x-axis. Now consider any point on the negative x-axis, say \( z = -1 \).

0.1. Convergence of functions. There are two notions of convergence, that of point-wise, and uniform convergence. We say that
- the sequence of functions \( f_n: \Omega \to \mathbb{C} \) converges point-wise to \( f: \Omega \to \mathbb{C} \), if for every \( z \in \Omega \), the sequence \( f_n(z) \to f(z) \). Or equivalently, given any \( \varepsilon > 0 \), and any \( z \in \Omega \), there exists an \( N > 0 \), possibly depending both on \( \varepsilon \) and \( z \), such that
  \[
n > N \implies |f_n(z) - f(z)| < \varepsilon.
\]
- the sequence of functions is said to converge uniformly if given any \( \varepsilon > 0 \), there exists an \( N > 0 \) depending only on \( \varepsilon \) such that for all \( z \in \Omega \) and \( n > N \) we have that
  \[
  |f_n(z) - f(z)| < \varepsilon.
\]
Theorem 0.3. If \( f_n : \Omega \to \mathbb{C} \) is a sequence of continuous functions which converge uniformly to \( f : \Omega \to \mathbb{C} \), then \( f \) itself is continuous.