LECTURE-3 : SOME TOPOLOGY

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Note that the distance between two complex numbers z, w is given by $|z-w|$. Given a $z_0 \in \mathbb{C}$, the *open* disc of radius R around z_0 is given by

$$D_R(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < R\}.$$

We now review a few standard definitions from topology. The *complement* of a set S, denoted by S^c is the set of all complex numbers NOT in S. Given any set $S \subset \mathbb{C}$, a point $p \in \mathbb{C}$ is a *limit* or an *accumulation* point if for any $r > 0$, the disc $D_r(p)$ has a point in common with S other than possibly p itself. The *closure* of a set S, denoted by $\bar{S}$ is the union of S with all it's accumulation points. The *interior* of S, denoted by $\mathring{S}$, is the set of all points $p \in S$ such that $D_r(p) \subset S$ for some $r > 0$. The *boundary* of a set S is the set of points $p \in \mathbb{C}$ such that for all $r > 0$, the disc $D_r(p)$ contains at least one point from S and S^c. For instance the boundary of the open disc $D_r(p)$ is the circle of radius r centered at p.

A set S is called *open* is for any point $p \in S$, there exists a disc $D_r(p) \subset S$. That is each point has a neighborhood that is completely contained in the set. A set is called *closed* is it's complement is open. An equivalent definition (why are they equivalent?) is that a set is closed if and only if it completely contains it's boundary. So for any set S, the interior $\mathring{S}$ is the largest open set contained in S and the closure $\bar{S}$ is the smallest closed set containing S. A set is called *compact* if it is closed and bounded. A basic property of open and closed sets is the following.

Proposition 0.1.
- *Arbitrary union (possibly infinite) of open sets is again open. Finite intersection of open sets is open.*
- *Arbitrary intersection of closed sets is close. Finite union of closed sets is closed.*

Given a sequence $\{z_n\}$ we say that it *converges* to $p \in \mathbb{C}$ if for all $\varepsilon > 0$, there exists an N such that

$$|z_n - p| < \varepsilon.$$

It is then easy to check the following.

Proposition 0.2. *$z_n \to p$ if and only if $Re(z_n) \to Re(p_n)$ and $Im(z_n) \to Im(p)$.*

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A disadvantage of this definition is that one needs to know the limit a priori, to even decide if a sequence is converging. A convenient alternative is the notion of a *Cauchy sequence*. A sequence z_n is said to be Cauchy if for all $\varepsilon > 0$, there exists an $N > 0$ such that for all $n, m > N$ we have

$$|z_n - z_m| < \varepsilon.$$

It is easy to see (prove it!) that every convergent sequence is Cauchy. Conversely, we have the following fundamental fact.

Theorem 0.1. *Every Cauchy sequence in $\mathbb{C}$ converges.*

In other words, $\mathbb{C}$ with the metric induced by $|\cdot|$ is a *complete metric space*. The theorem follows from the proposition above and the fact that real numbers from a complete metric space. A consequence of completeness is the following useful characterization of compact sets in $\mathbb{C}$.

Theorem 0.2. *The following are equivalent.*

(1) *$K \subset \mathbb{C}$ is compact.*
(2) *Any infinite sequence $\{z_n\} \subset K$ has an accumulation point $p \in K$.*

Proof. $1 \implies 2$. Assume K is compact, and let $\{z_n\}$ be any infinite sequence. Since K is bounded, we can enclose it in a large square S. Subdivide the square into four equal parts. Since the sequence is infinite, at least one of the sub-squares must also have infinitely many points from $\{z_n\}$. Call this sub-square S_1. Continuing this procedure, we have a sequence of squares S_k such that

- The S_k from a decreasing sequence, that is $S_k \subset S_{k-1}$ for all $k > 0$.
- Each S_k has infinitly many points from the sequence $\{z_n\}$.
- S_k has side length half of that of S_{k-1}.

Pick a point $z_{n_k} \in S_k$. Since the side lengths go to zero, and the fact that S_k are nested and decreasing, the sub-sequence $\{z_{n_k}\}$ forms a Cauchy sequence. By completeness of $\mathbb{C}$ it must converge to a limit p. Clearly p is an accumulation point. This is because any disc around it will contain one of the squares S_k, and hence must contain infinitely many points from the sequence $\{z_n\}$. In particular any such disc must contain at least one point of K different from p. But then since K is closed, it must contain all it's accumulation points, and hence $p \in K$. there by completing the proof.

$2 \implies 1$. This is easy. We need to show that K is closed and bounded. If it is not bounded, we could choose a sequence $\{z_n\}$ such that $|z_n| \to \infty$. This cannot have an accumulation point, contradicting 2. Hence K has to be bounded. If it is not closed, then there is an accumulation point $p \notin K$. But then consider the discs $D_{1/n}(p)$ of radius $1/n$ around p. Since p is an accumulation point, we can form a sequence of points $\{z_n\}$ distinct from p such that $z_n \in D_{1/n}(p) \cap K$. Clearly the only accumulation point of the sequence is p. But p does not belong to K, which contradicts 2. Hence K has to be closed.

$\square$

In the course of the proof, we actually prove the following property of closed sets, which is interesting in it's own right.

Lemma 0.1. *Let K_n be a nested sequence of closed sets, that is*

$$\cdots K_n \subset K_{n-1} \cdots \subset K_1,$$

such that

$$\lim_{n\to\infty} \operatorname{diam}(K_n) = 0,$$

then the intersection

$$\cap_{n=1}^{\infty} K_n,$$

which is closed set, consists of exactly one point.

The last notion we need is that of a connected set. A subset $S \subset \mathbb{C}$ is called connected, if

$$S = (U \cap S) \cup (V \cap S)$$

for some disjoint open sets U and V. If S itself is open, this reduces to saying that S cannot be written as the union of two disjoint open sets. An open, connected subset is called a *region*. In the next lecture, we will see that a subset is a region if and only if any two points can be connected by a continuous curve.