

LECTURE-2 : THE COMPLEX PLANE

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It is known that certain polynomial equations with real coefficients need not have real roots. Complex numbers are obtained from the reals by formally adjoining a number i that solves the equation

$$i^2 = -1.$$

So a general complex number takes the form

$$z = x + iy,$$

where x and y are real numbers, and are called the real and imaginary part of z respectively. We use the notations $x = \operatorname{Re}(z)$ and $y = \operatorname{Im}(z)$. The set of complex numbers are denoted by \mathbb{C} . Clearly the real numbers can be identified as a subset of the complex numbers in a natural way as numbers with $\operatorname{Im}(z) = 0$. We define addition and subtraction to be component-wise i.e. if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then we define

$$z_1 \pm z_2 = (x_1 + x_2) \pm i(y_1 + y_2).$$

Using this, we can identify \mathbb{C} with \mathbb{R}^2 as vector spaces. With this interpretation, a complex number represents a point in the xy -plane; with the x -coordinate given by $\operatorname{Re}(z)$ and the y -coordinate given by $\operatorname{Im}(z)$. This more geometric interpretation will be very useful to us.

But the complex numbers are much more than just 2-dimensional vectors. They also have a multiplicative structure. That is we can multiply two complex numbers to obtain another complex number. This is different from the dot or cross product of two dimensional vectors; the first produces a scalar (ie. a real number) and the second produces a vector perpendicular to the xy -plane. Multiplication can be defined simply by mimicking the distribution law and remembering that $i^2 = -1$. So if z_1 and z_2 are as above, we define

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1).$$

It is then easy to verify the following properties for addition and multiplication.

P1 (Additive and Multiplicative identity.) For any complex number z ,

$$z + 0 = z, \quad z \cdot 1 = z.$$

P2 (Commutativity.) For any $z_1, z_2 \in \mathbb{C}$,

$$z_1 + z_2 = z_2 + z_1, \quad z_1 z_2 = z_2 z_1.$$

Date: 24 August 2016.

P3 (Associativity.) For any complex numbers z_1, z_2, z_3 ,

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3), (z_1 z_2) z_3 = z_1 (z_2 z_3).$$

P4 (Distribution) For any $z_1, z_2, z_3 \in \mathbb{C}$,

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3.$$

P5 (Additive inverse.) For any $z \in \mathbb{C}$, $-z = (-1) \cdot z$ satisfies

$$z + (-z) = 0.$$

As remarked above, geometrically, addition of complex numbers corresponds to addition of vectors. What is the interpretation for multiplication? This is clearer if we use *polar coordinates*. Recall that any point (x, y) in the plane that is not the origin, can be represented uniquely by a pair (r, θ) , where $r > 0$ and $\theta \in (-\pi, \pi]$ such that

$$x = r \cos \theta, y = r \sin \theta.$$

Then r is the geometric distance from the origin, and θ is the angle made by the line joining (x, y) to the origin with the positive x -axis. For instance the complex number i corresponds to $(1, \pi/2)$ in polar coordinates. So, any complex number can be represented as

$$z = r(\cos \theta + i \sin \theta).$$

If $w = \rho(\cos \alpha + i \sin \alpha)$ is another complex number, then it follows from the definition of the multiplication formula that

$$\begin{aligned} zw &= r\rho[(\cos \theta \cos \alpha - \sin \theta \sin \alpha) + i(\cos \theta \sin \alpha + \sin \theta \cos \alpha)] \\ &= r\rho(\cos(\theta + \alpha) + i \sin(\theta + \alpha)), \end{aligned}$$

where we used the sum-angle formulas in the last equation. So geometrically multiplication simply corresponds to a dilation (i.e. scaling) and a rotation. For instance multiplication by i corresponds to rotating the vector representing the complex number by $\pi/2$. To form a good number system we will also need to be able to divide by complex numbers. For any $z_2 \neq 0$, we say that $w = z_1/z_2$ if $z_1 = wz_2$. If z_2 is a real number then the quotient is simply by dividing the real and imaginary parts of z_1 by z_2 . In general, a formula for division in terms of the real and imaginary parts is most conveniently expressed using the modulus and conjugate. We define these next.

Conjugate, Absolute value, Argument. To define division, it is enough to have a formula for $1/z$. Let $z = x + iy$. Multiplying the numerator and denominator of $1/z$ by $x - iy$ (this is similar to rationalizing irrational denominators) we obtain

$$\frac{1}{z} = \frac{1}{x + iy} = \frac{x - iy}{(x + iy)(x - iy)} = \frac{x - iy}{x^2 + y^2}.$$

Note that in the last equation, the denominator is now a real number, and we already know how to divide by real non-zero numbers. The numerator, is called the *conjugate* of z and is denoted by

$$\bar{z} = x - iy.$$

Geometrically this amounts to reflection of the point representing z about the x -axis. Readers will notice that the denominator is the square of the distance of the point (x, y) from the origin. So we define the absolute value or the *length* of the complex number, denote by $|z|$ as

$$|z| = \sqrt{x^2 + y^2}.$$

This is of course the ‘ r ’ in the polar coordinate representation. Some basic properties of these operations are the following.

- $\overline{\bar{z}} = z$.
- $|\bar{z}| = |z|$.
- $|z| = 0 \implies z = 0$.
- $\overline{z\bar{w}} = \bar{z}w$, $|zw| = |z||w|$.
- $\overline{z + w} = \bar{z} + \bar{w}$.

Note that contrary to the conjugate function, the modulus function is not additive. Instead we have an inequality; see Theorem 0.1. With these notations in place, we can re-write the above statement as

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2},$$

for any $z \neq 0$. So we in summary we shown that multiplication has another property, that every no-zero number has *multiplicative inverse*. That is we have

P6 (multiplicative inverse.) For every $z \in \mathbb{C}$, $z \neq 0$ there exists a complex number $1/z = \bar{z}/|z|^2$ such that

$$z \cdot \frac{1}{z} = 1.$$

With the two operations of addition and multiplication satisfying these six axioms, the set of complex numbers become what is called as *field* by algebraists. In fact, the complex numbers form an *algebraically closed* field, which means that any polynomial with complex coefficients can be completely factorized using complex roots. Later in the course, somewhat remarkably, we will prove this statement in algebra using our complex analysis techniques. We will in fact give multiple proofs, not just one!

The ‘ θ ’ in the polar coordinates also has a name, and is called the *argument* of z , and denote by $\arg(z)$. Using the new notation, it is also easy to see that

$$Re(z) = \frac{z + \bar{z}}{2}, \quad Im(z) = \frac{z - \bar{z}}{2i}.$$

We can now define division by

$$\frac{z}{w} = \frac{z\bar{w}}{|w|^2}.$$

Integer Powers. Given any natural number $n \in \mathbb{N}$, we define z^n to be z multiplied to itself ' n ' times. We also define $z^0 := 1$. For negative integers $-n$, we then define z^{-n} to be $1/z^n$ or the multiplicative inverse to z^n .

We end with an important inequality that will be crucial in most of the estimates.

Theorem 0.1 (Triangle inequality). *Let $z, w \in \mathbb{C}$. Then we have the following inequalities*

(1)

$$|z + w| \leq |z| + |w|,$$

with equality if and only if $z = aw$ where $a \in \mathbb{R}$ i.e. z and w lie on the same line through the origin.

(2)

$$|z - w| \geq ||w| - |z||,$$

with equality again if z and w lie on the same line through the origin.

Proof. To see the first inequality, consider a triangle whose vertices are the origin O , the point represented by z , which we call A and the point represented by $z + w$ which we label B . Then by triangle inequality from elementary geometry, every side is smaller than the sum of the other two sides with equality only if all three points lie on the same line. So if they don't lie on the same line, we have

$$|OB| < |OA| + |AB|.$$

The proof is completed by noticing that $|OB| = |z + w|$, $|OA| = |z|$ and $|AB| = |w|$. For the second inequality, without loss of generality, suppose $|w| \geq |z|$. Then the right hand side is $|w| - |z|$. So applying the first inequality, since $w = (w - z) + z$,

$$|w| \leq |w - z| + |z|,$$

from which the desired inequality follows with equality if and only if $w - z$ and z lie on the same straight line. But then this happens if and only if z and w themselves lie on the same line through the origin. \square

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