Residue Calculus - II

— Integration along branch cut.

• **Type IV**: $\displaystyle\int_0^\infty x^\alpha R(x)\,dx$, $\alpha \in (0,1)$, R rational function.

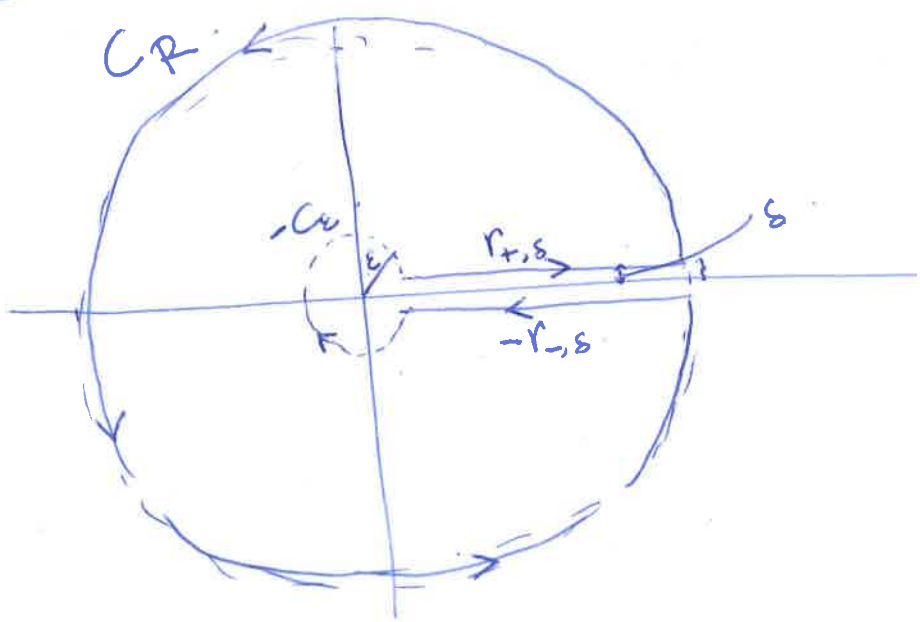
Assumption: $R(x) = P(x)/Q(x)$. s.t

1) $\deg Q \geq \deg P + 2$.
2) $Q(0) = 0 \Rightarrow 0$ is a simple root of Q.
   i.e $\lim_{z\to 0} Q(z)/z \neq 0$.

Assumption $\Longrightarrow$ integral is convergent by p-test

$$\Longrightarrow \int_0^\infty x^\alpha R(x)\,dx = \lim_{R\to\infty} \int_0^R x^\alpha R(x)\,dx.$$

Contour:



We try to integrate the function $z^\alpha R(z)$ on the contour.

<u>Branch</u>: Recall $z^\alpha = e^{\alpha \log z}$; and we choose the <u>branch</u> of $\log z$ s.t. if $z = re^{i\theta}$.

$$\log z = \log|z| + i \theta.$$

with $\theta \in (0, 2\pi)$. <u>Note</u> this is diff from the principal branch where arg was used (i.e $\theta \in (-\pi, \pi)$).

<u>Lemma</u>:
$$\lim_{\delta \to 0} \int_{\Gamma_{+,\delta}} z^\alpha R(z)\,dz = \int_\varepsilon^R x^\alpha R(x)\,dx.$$

$$\lim_{\delta \to 0} \int_{\Gamma_{-,\delta}} z^\alpha R(z)\,dz = e^{2\pi i \alpha} \int_\varepsilon^R x^\alpha R(x)\,dx.$$

The idea is that on $\Gamma_{-,\delta}$, $\left.\begin{array}{l} \theta \approx 2\pi, \\ |z| \approx x. \end{array}\right\}$ so $z^\alpha \approx e^{2\pi i \alpha} \cdot x^\alpha$

From, now we denote
$$\lim_{\delta \to 0} \int_{\Gamma_{+,\delta}} z^\alpha R(z)\,dz = \int_{\Gamma_+} z^\alpha R(z)\,dz$$

$$\lim_{\delta \to 0} \int_{\Gamma_{-,\delta}} z^\alpha R(z)\,dz = \int_{\Gamma_-} z^\alpha R(z)\,dz$$

as the integrals on upper and lower branches respectively. By residue theorem

$$\int_{C_R} z^\alpha R(z)\,dz - \int_{\gamma_-} z^\alpha R(z)\,dz - \int_{C_\varepsilon} z^\alpha R(z)\,dz$$

$$+ \int_{\gamma_+} z^\alpha R(z)\,dz = 2\pi i \sum_{\substack{Q(\beta)=0 \\ \beta \neq 0}} \operatorname{Res}_{z=\beta} z^\alpha R(z)$$

But $$\left(\int_{\gamma_+} - \int_{\gamma_-}\right) z^\alpha R(z)\,dz = \left(1 - e^{2\pi i \alpha}\right) \int_\varepsilon^R R(x) \cdot x^\alpha\,dx.$$

$$\Rightarrow \left(1 - e^{2\pi i\alpha}\right) \int_\varepsilon^R R(x)\, x^\alpha\,dx = \left(-\int_{C_R} + \int_{C_\varepsilon}\right) R(z) z^\alpha\,dz$$

$$+ 2\pi i \sum_{Q(\beta)=0} \operatorname{Res}_{z=\beta} z^\alpha R(z)$$

Letting $\varepsilon \to 0$, $R \to \infty$, we need to show that the integrals on the right go to zero.

Example: $a \in (0,1)$, $$\int_0^\infty \frac{x^{-a}}{1+x}\,dx = \int_0^\infty \frac{x^{1-a}}{x(1+x)}\,dx.$$ with $\alpha = 1-a$

By the previous argument,

$$\left(1 - e^{2\pi i(1-a)}\right) \int_0^\infty \frac{x^{-a}}{1+x}\,dx = -\lim_{R\to\infty} \int_{C_R} \frac{z^\alpha}{z(1+z)}\,dz$$

$$(*) \qquad + \lim_{\varepsilon \to 0} \int_{C_\varepsilon} \frac{z^\alpha}{z(1+z)}\,dz$$

$$+ \; 2\pi i \operatorname{Res}_{z=-1} \frac{z^\alpha}{z(1+z)}.$$

- <u>Integral on C_R</u>

$$z^\alpha = e^{\alpha \log z}.$$

$$\Rightarrow \quad |z^\alpha| = |e^{\alpha \log z}| = e^{\alpha \log|z|} = |z|^\alpha.$$

So
$$\left|\frac{z^\alpha}{z(1+z)}\right| = \frac{|z|^\alpha}{|z|\,|1+z|}$$

If R is big, $|1+z| > |z|/2$ if $z \in C_R$.

$$\Rightarrow \quad \left|\frac{z^\alpha}{z(1+z)}\right| \le \frac{2}{|z|^{2-\alpha}} \le \frac{2}{R^{2-\alpha}}$$

$$\Rightarrow \quad \left|\int_{C_R} \frac{z^\alpha}{z(1+z)}\,dz\right| < \frac{2}{R^{2-\alpha}} \cdot \operatorname{len}(C_R)$$

$$< \frac{2}{R^{2-\alpha}} 2\pi R = \frac{4\pi}{R^{1-\alpha}} \xrightarrow{R\to\infty} 0$$

since $\alpha < 1$.

- <u>Integral on C_ε</u>: On C_ε, $|1+z| > 1/2$ if ε is small.

So
$$\left|\frac{z^\alpha}{z(1+z)}\right| \le \frac{2}{|z|^{1-\alpha}} \le \frac{2}{\varepsilon^{1-\alpha}}.$$

$$\rightarrow \quad \left|\int_{C_\varepsilon} \frac{z^\alpha}{z(1+z)}\,dz\right| \le \frac{2}{\varepsilon^{1-\alpha}} \operatorname{len}(C_\varepsilon) = \frac{4\pi\varepsilon}{\varepsilon^{1-\alpha}}$$

$$= 4\pi\varepsilon^\alpha \xrightarrow{\varepsilon\to 0} 0, \quad \text{since } \alpha > 0.$$

~~So we let~~

• Residue: $\operatorname{Res}_{z=-1} \dfrac{z^{\alpha}}{z(1+z)} = \lim\limits_{z\to -1} \dfrac{(1+z)\, z^{\alpha}}{(1+z)\, z}$

$$= \lim_{z\to -1} z^{\alpha-1} = (-1)^{-a} \qquad \text{since } 1-\alpha = a.$$

With the branch cut chosen, $\log(-1) = i\pi$ and so

$$\operatorname{Res}_{z=-1} \frac{z^{\alpha}}{z(1+z)} = e^{-i\pi a}.$$

So from $(*)$,

$$\int_0^{\infty} \frac{x^{-a}}{1+x}\, dx = \frac{2\pi i\, e^{-\pi i a}}{1 - e^{2\pi i(1-a)}} = \frac{2\pi i\, e^{-i\pi a}}{1 - e^{-2\pi i a}}$$

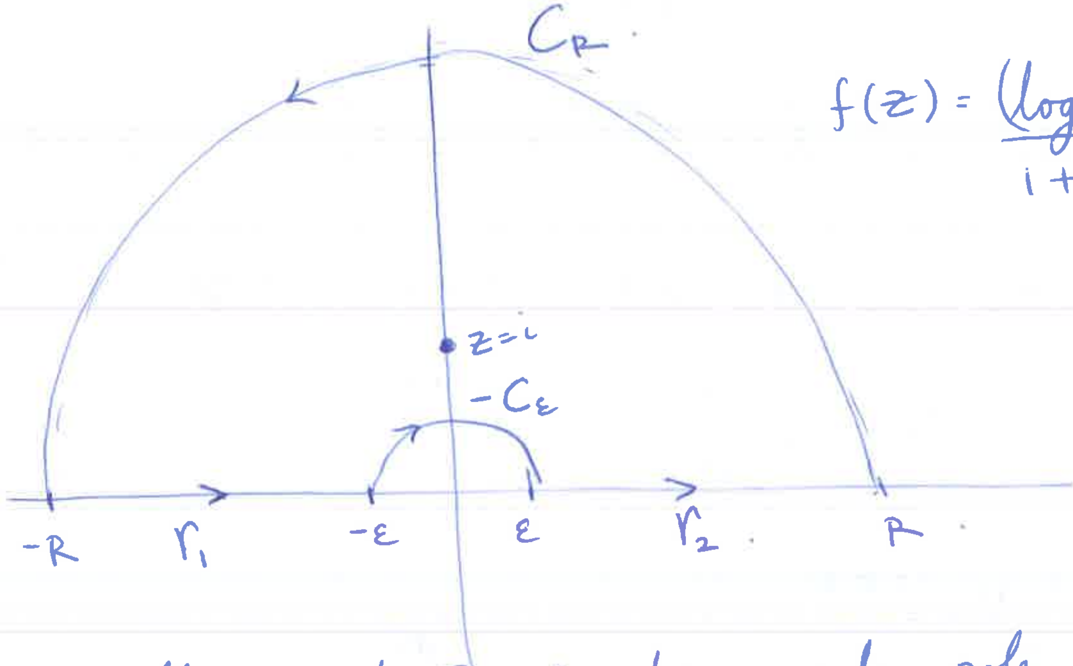
$$= \pi \frac{2i}{e^{i\pi a} - e^{-i\pi a}} = \boxed{\pi / \sin \pi a}$$

$a \in (0,1)$.

• By using a similar ~~clever~~ choice of contour one can also evaluate integrals involving $\log x$. One can also use other contours such as semicircles.

Example: $I = \displaystyle\int_0^{\infty} \frac{(\log x)^2}{1+x^2}\, dx$

We invite the reader to use the contour from previous problem. We instead use



$$f(z) = \frac{(\log z)^2}{1+z^2}.$$

If ε is small, and R is big, only pole at $z=i$.

<u>Branch</u>: To use the contour, we let $z = re^{i\theta}$,

$$\log z = \log r + i\theta$$

where $\boxed{-\pi/2 < \theta < 3\pi/2}$

Clearly

$$\int_{\gamma_1} f(z)\,dz \xrightarrow[\varepsilon \to 0]{R \to \infty} \int_0^\infty \frac{(\ln x)^2}{1+x^2}\,dx = I$$

$$\int_{\gamma_2} f(z)\,dz \xrightarrow[\varepsilon \to 0]{R \to \infty} \int_{-\infty}^{0} \frac{(\ln|t| + i\pi)^2}{1+t^2}\,dt.$$

parametrize, $\gamma_2(t) = t$

Put $-t = x$, Then

$$\int_{\Gamma_2} f(z)\,dz \xrightarrow[\varepsilon \to 0]{R \to \infty} \int_0^\infty \frac{(\ln x + \pi i)^2}{1+x^2}\,dx$$

$$= \int_0^\infty \frac{(\ln x)^2}{1+x^2}\,dx + 2\pi i \int_0^\infty \frac{\ln x}{1+x^2}\,dx$$

$$-\pi^2 \int_0^\infty \frac{dx}{1+x^2}.$$

$$= I + 2\pi i \int_0^\infty \frac{\ln x}{1+x^2}\,dx - \pi^2 \int_0^\infty \frac{dx}{1+x^2}.$$

Note that the second integral is purely imaginary: So

$$\operatorname{Re}\left(\lim_{R\to\infty} \lim_{\varepsilon\to 0} \left[\int_{\Gamma_1} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz\right]\right) = 2I - \pi^2 \int_0^\infty \frac{dx}{1+x^2}. \quad (**)$$

By analysis similar to the previous example, it can be shown that

$$\lim_{R\to\infty} \int_{C_R} f(z)\,dz = \lim_{\varepsilon\to 0} \int_{C_\varepsilon} f(z)\,dz = 0.$$

So, by the residue theorem

$$\operatorname{Re}\left(\lim_{R\to\infty}\lim_{\varepsilon\to 0}\int_{\gamma_1} + \int_{\gamma_2}\right) = \operatorname{Re}\left(2\pi i \operatorname{Res}_{z=i} f(z)\right).$$

$$\operatorname{Res}_{z=i} f(z) = \lim_{z\to i}\frac{(z-i)(\log z)^2}{1+z^2} = \lim_{z\to i}\frac{(\log z)^2}{z+i} = \frac{(\log i)^2}{2i}$$

With the branch chosen, $\log i = i\pi/2$.

So $\operatorname{Res}_{z=i} f(z) = -\pi^2/8i$

$$\Rightarrow \quad \operatorname{Re}\left(2\pi i \operatorname{Res}_{z=i} f(z)\right) = -\pi^3/4.$$

From (**)

$$2I - \pi^2\int_0^\infty \frac{dx}{1+x^2} = -\pi^3/4.$$

From Calc-2, $\displaystyle\int_0^\infty \frac{dx}{1+x^2} = \arctan x\Big|_0^\infty = \pi/2.$

$$\Rightarrow \quad 2I = \pi^3/2 - \pi^3/4 \Rightarrow \boxed{I = \pi^3/8}$$

Rk: Taking the imaginary parts, we also obtain the formula

$$\int_0^\infty \frac{\ln x}{1+x^2}\,dx = 0.$$

which is not completely obvious! Think of an elementary reason!