Residue Calculus - I

The purpose of this section is to apply residue theorem to evaluating real integrals.

Recall the residue theorem. If $f : \Omega \to \mathbb{C}$ is meromorphic, $\gamma \subset \Omega$ is simple, closed s.t $Int(\gamma) \subset \Omega$. If $p_1, \dots p_N \in Int(\gamma)$ are poles of f s.t f is hol on $Int(\gamma)/\{p_1, \dots p_N\}$, & $p_j \notin \gamma$ $\forall j$, then

$$\boxed{\int_\gamma f(z)\,dz = 2\pi i \sum_{j=1}^{N} Res_{p_j} f(z)}$$

We will learn to evaluate five types of integrals.

• Type 1

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\,d\theta, \quad R \text{ rational function}$$

Assumption: No pole for $\theta \in [0, 2\pi]$.

Method: Consider $z = z(\theta) = e^{i\theta}$

Then
$$\begin{cases} \cos\theta = \dfrac{z + \bar{z}}{2} = \dfrac{z + 1/z}{2} \\ \sin\theta = \dfrac{z - \bar{z}}{2i} = \dfrac{z - 1/z}{2i} \end{cases}$$

$$z'(\theta) = ie^{i\theta}d\theta$$

$$\Longleftrightarrow \quad d\theta = -i\,dz/z.$$

So, we can transform the integral to

$$-i \int_{|z|=1} R\left(\frac{z+1/z}{2}, \frac{z-1/z}{2i}\right)\frac{dz}{z}$$

Example: $\displaystyle\int_0^{\pi} \frac{d\theta}{a+\cos\theta}, \quad a>1$

Observe: since $\cos(\theta+\pi) = \cos\theta$

$$\int_{\pi}^{2\pi} \frac{d\theta}{a+\cos\theta} = \int_0^{\pi} \frac{d\theta}{a+\cos\theta}$$

So if we call the integral that we need by I, then

$$I = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{a+\cos\theta}$$

letting $z = e^{i\theta}$, noting that $\bar{z} = 1/z$ since $|z|=1$, we see that

$$\cos\theta = \frac{z+1/z}{2}$$

So

$$\int_0^{2\pi} \frac{d\theta}{a+\cos\theta} = -i\int_{|z|=1} \frac{1}{a+(z+1/z)/2}\cdot\frac{dz}{z}$$

$$= -2i\int_{|z|=1} \frac{dz}{z^2+2az+1}$$

So,

$$I = -i\int_{|z|=1} \frac{dz}{z^2+2az+1}$$

Now, $z^2+2az+1=0$ has 2 solutions.

$$\alpha = -a+\sqrt{a^2-1}, \quad \beta = -a-\sqrt{a^2-1}.$$

Since $a>1$, these are real. Moreover, $|\alpha|<1$ & $|\beta|>1$. So,

$$I = -i\int_{|z|=1} \frac{dz}{z^2+2az+1} = (-i)(2\pi i)\operatorname{Res}_{z=\alpha}\frac{1}{z^2+2az+1}$$

$$= 2\pi \operatorname{Res}_{z=\alpha}\frac{1}{z^2+2az+1}$$

$$= 2\pi \lim_{z\to\alpha}\frac{z-\alpha}{z^2+2az+1}$$

$$= 2\pi \lim_{z\to\alpha}\frac{1}{z-\beta} = \frac{2\pi}{\alpha-\beta}$$

$$= \boxed{\frac{\pi}{\sqrt{a^2-1}}}.$$

<u>Type II</u> $\int_{-\infty}^{\infty} R(x)\,dx$, R rational.

<u>Assumption</u>: $R(x) = P(x)/Q(x)$.

$\deg Q \geq \deg P + 2$.

Q has no <u>real root</u>

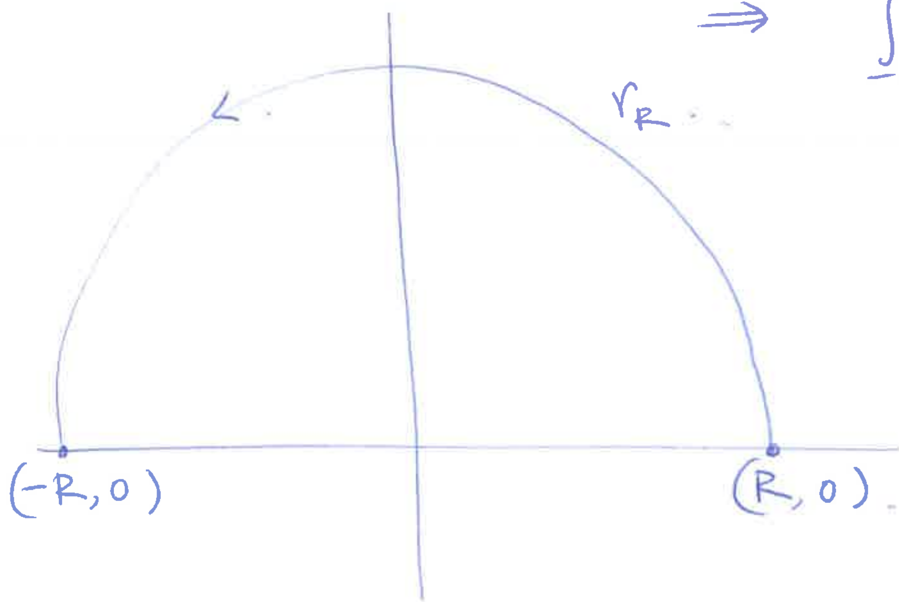
<u>Recall</u>:
$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R_1,R_2\to\infty}\int_{-R_1}^{R_2} f(x)\,dx$$

if f is cont on $\mathbb{R}$.

If limit exists, we say integral is convergent.
If integral is convergent, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx.$$

Method: Use semicircle. $\deg Q \geq \deg P + 2$
$\Rightarrow \int_{-\infty}^{\infty} R(x)\,dx$ is conv.



If R is big enough to contain all roots α of Q s.t. $\operatorname{Im}\alpha > 0$, then by residue theorem.

$$\int_{-R}^{R} R(x)\,dx + \int_{\gamma_R} R(z)\,dz = 2\pi i \sum_{\substack{Q(\alpha)=0 \\ \operatorname{Im}\alpha>0}} \operatorname{Res}_{z=\alpha} R(z).$$

Since $\deg Q \geq \deg P + 2$, on γ_R,

$$|R(z)| \leq \frac{M}{|z|^2} = \frac{M}{R^2}.$$

$$\Rightarrow \left| \int_{\gamma_R} R(z)\,dz \right| \leq \frac{M}{R^2} \cdot \pi R \leq \frac{M\pi}{R} \xrightarrow{R\to\infty} 0$$

Taking $R \longrightarrow \infty$.

$$\int_{-\infty}^{\infty} R(x)\,dx = 2\pi i \sum_{\substack{Q(\alpha)=0 \\ \operatorname{Im}\alpha>0}} \operatorname{Res}_{z=\alpha} R(z).$$

Example: $\displaystyle\int_0^{\infty} \frac{x^2}{1+x^6}\,dx$.

Since function is even.

$$I = \int_0^{\infty} \frac{x^2\,dx}{1+x^6} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2\,dx}{1+x^6}.$$

By the above reasoning

$$I = \pi i \sum_{\substack{\alpha \text{ s.t.} \\ 1+\alpha^6=0 \\ \operatorname{Im}\alpha>0}} \operatorname{Res}_{\alpha} \frac{z^2}{1+z^6}.$$

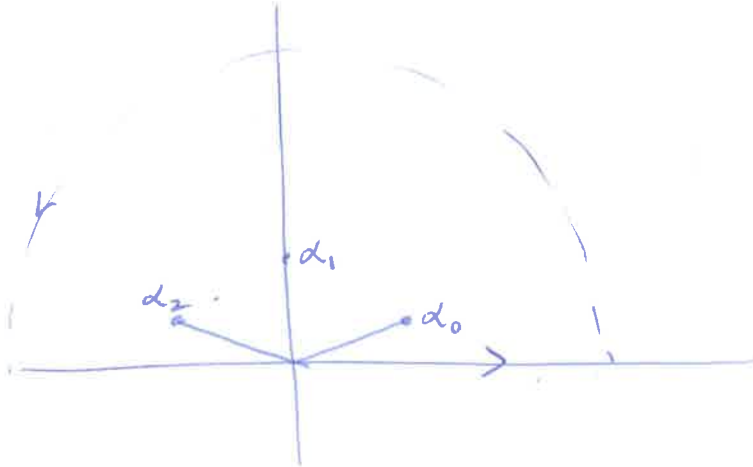
Now, the roots of $1+x^6=0$ are given by

$$\alpha_k = e^{i(\pi/6 + 2\pi i k/6)}, \quad k=0,1,\dots,5.$$

Of these, only

$$\alpha_0 = e^{i\pi/6}, \quad \alpha_1 = i, \quad \alpha_2 = e^{5i\pi/6}$$

satisfy $\operatorname{Im}\alpha > 0$.



To find residue, since poles are simple

$$\mathrm{Res}_{z=\alpha_k} \frac{z^2}{1+z^6} = \alpha_k^2 \lim_{z\to\alpha_k} \frac{z-\alpha_k}{1+z^6}$$

$$\underset{\text{L'Hopital}}{=} \alpha_k^2 \lim_{z\to\alpha_k} \frac{1}{6z^5} = \frac{1}{6\alpha_k^3}$$

Let $A_k = \mathrm{Res}_{z=\alpha_k} z^2/1+z^6$, then

$$A_0 = \frac{e^{-\pi i/2}}{6} = -\frac{i}{6}$$

$$A_1 = i/6$$

$$A_2 = -i/6$$

$A_0 + A_1 + A_2 = -i/6$.

So

$$I = \pi i \cdot \left(\frac{-i}{6}\right) = \pi/6.$$

Type III

$$\int_{-\infty}^{\infty} R(x)\, e^{ix}\, dx \quad \begin{cases} \displaystyle\int_{-\infty}^{\infty} R(x) \sin x \, dx \\ \displaystyle\int_{-\infty}^{\infty} R(x) \cos x \, dx . \end{cases}$$

Type III(a): $R(x) = P(x)/Q(x)$, $\deg Q \geq \deg P + 2$. no ~~all~~ root of Q is real.

Again use semi-circle, and proceed as before. Key point is that integral is convergent by p-test.

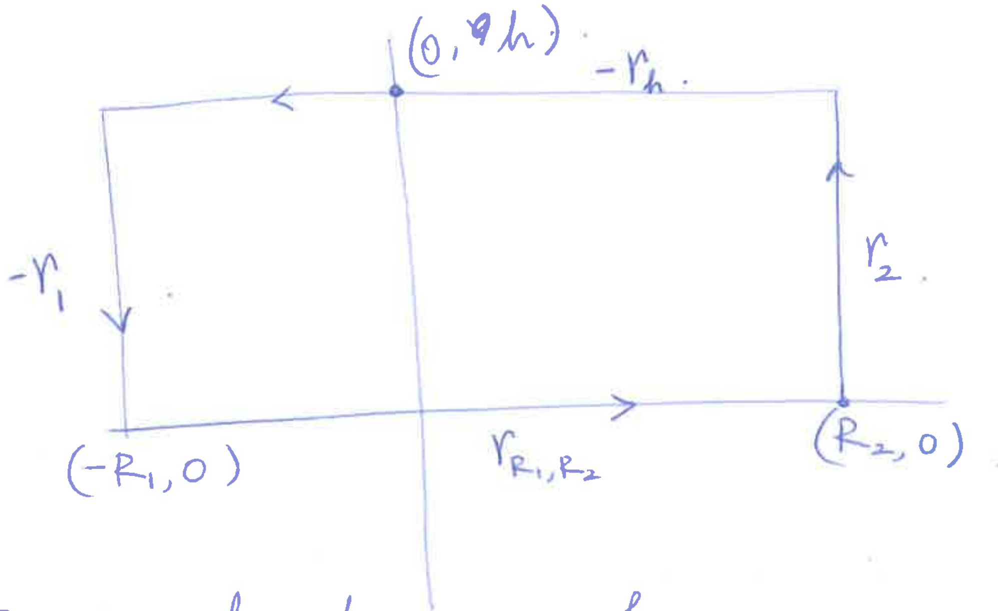
Type III(b): $R(x) = P(x)/Q(x)$, $\deg Q = \deg P + 1$.

CASE 1: No real root of Q.

It is no longer clear that the integral converges. So cannot use semi-circle, since

$$\int_{-\infty}^{\infty} R(x)\, e^{ix}\, dx = \lim_{R_1, R_2 \to \infty} \int_{-R_1}^{R_2} R(x)\, e^{ix}\, dx$$

where R_1, R_2 could be different. So have to use rectangle.



If R_1, R_2, h big enough

$$\left(-\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_{R_1,R_2}} - \int_{\gamma_h} \right) R(z) e^{iz} \, dz$$

$$= 2\pi i \sum_{\substack{Q(\alpha)=0 \\ \operatorname{Im}\alpha > 0}} \operatorname{Res}_\alpha R(z) e^{iz}$$

where paths are given +ve orientation if they go from bottom to top or left to right (hence −ve for $\int_{\gamma_1}$ & $\int_{\gamma_h}$).

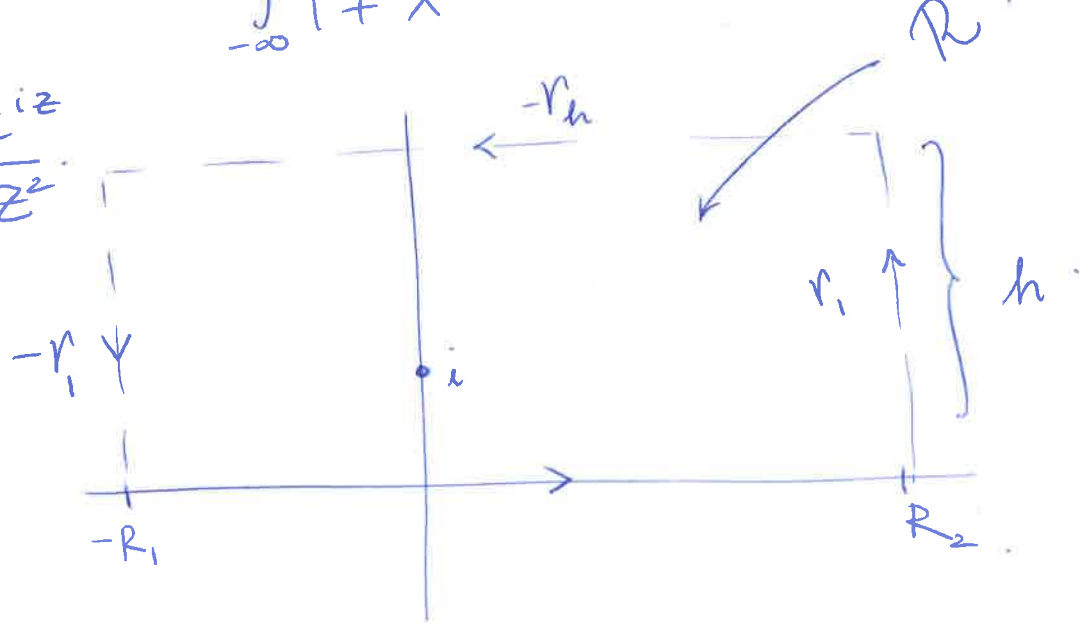
• Integrals $\int_{\gamma_1}$, $\int_{\gamma_2}$.

Consider γ_2. Since $\deg Q = \deg P + 1$, on $|z| > 1$, $|R(z)| \leq \dfrac{M}{|z|}$, for some $M > 0$.

The idea is to first fix R_1, R_2 and let $h \to \infty$ and then let $R_1, R_2 \to \infty$. Letting $h \to \infty$ will make $\int_{\gamma_h} \to 0$ while letting will let $\int_{\gamma_1}, \int_{\gamma_2} \to 0$. To illustrate this, consider the foll. example

<u>Example</u>: $\displaystyle\int_{-\infty}^{\infty} \frac{x e^{ix}}{1+x^2}\,dx$

$$f(z) = \frac{z e^{iz}}{1+z^2}$$



<u>If</u> h is big enough, rectangle R contain the ~~only~~ pole $z = i$.

$$\operatorname{Res}_{z=i} f(z) = \lim_{z \to i} (z-i) f(z)$$

$$= \lim_{z \to i} \frac{z e^{iz} \cancel{(z-i)}}{(z+i)\cancel{(z-i)}}$$

$$= \frac{1}{2e}.$$

By the residue theorem

$$\left(\int_{\gamma_{R_1,R_2}} + \int_{\gamma_2} - \int_{\gamma_h} - \int_{\gamma_1} \right) \frac{z e^{iz}}{1+z^2} dz = 2\pi i \cdot \frac{1}{2e}$$

$$= \boxed{\frac{\pi i}{e}}$$

Analyze $\int_{\gamma_h}$. On γ_h, $|z| > h \Rightarrow \left|\frac{z}{1+z^2}\right| \leq \frac{C}{h}$ for some $C > 0$.

$$|f(z)| = \left| \frac{z e^{iz}}{1+z^2} \right| \leq \frac{C}{h} \cdot |e^{iz}|.$$

But $|e^{iz}| = e^{-\mathrm{Im}\, z} = e^{-h}$.

$\Rightarrow$ On γ_h, $|f(z)| \leq Ce^{-h}/h$.

So,

$$\left| \int_{\gamma_h} \frac{z e^{iz}}{1+z^2} dz \right| \leq \frac{C e^{-h}}{h} \operatorname{len}(\gamma_h) = \frac{Ce^{-h}(R_1+R_2)}{h} \xrightarrow{h \to \infty} 0$$

Analyze $\int_{\gamma_2} f(z)\,dz$. On γ_2, $|z| > R_2$.

$\Rightarrow \left|\frac{z}{1+z^2}\right| \leq \frac{2}{R_2}$ if R_2 is big enough.

$$\Rightarrow \left| \int_{\gamma_2} \frac{z e^{iz}}{1+z^2} dz \right| \leq \frac{2}{R_2} \int_{\gamma_2} e^{-\mathrm{Im}\, z} \cdot |dz|$$

parametrize $\left(\begin{array}{l} \gamma_2(t) = R_2 + it \\ |dz| = dt \end{array} \right)$

$$= \frac{2}{R_2} \underbrace{\int_0^h e^{-t} dt}_{\leq 1} < \frac{2}{R_2}.$$

$\Rightarrow$ if R_1 is big enough

$$\left| \int_{\Gamma_1} \frac{e^{iz} z}{1+z^2}\, dz \right| \leq \frac{2}{R_1}.$$

Similarly
$$\left| \int_{\Gamma_2} \frac{e^{iz} z\, dz}{1+z^2} \right| \leq \frac{2}{R_2}.$$

So, fixing R_1, R_2 and letting $h \to \infty$.

$$\left| \int_{-R_1}^{R_2} \frac{x e^{ix}}{1+x^2}\, dx - \frac{\pi i}{e} \right| \leq 2\left(\frac{1}{R_1} + \frac{1}{R_2} \right) \xrightarrow{R_1, R_2 \to \infty} 0.$$

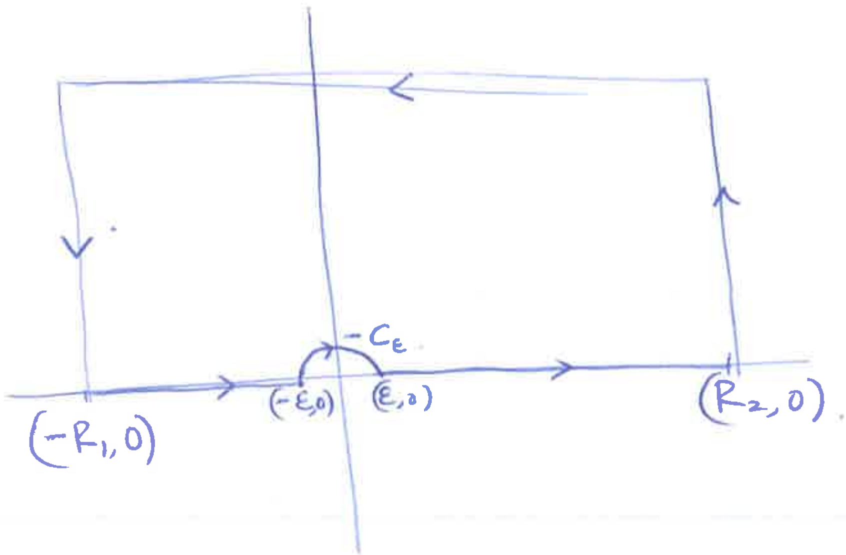
$$\Rightarrow \boxed{\int_{-\infty}^{\infty} \frac{x e^{ix}\, dx}{1+x^2} = \frac{\pi i}{e}.}$$

• <u>CASE 2</u>: $R(x)$ has a simple pole on real axis at one of the zeroes of $\sin x$ or $\cos x$. ~~Then~~ (but not both).

Then even though $\int_{-\infty}^{\infty} R(x) e^{ix}\, dx$ will not exist, $\int_{-\infty}^{\infty} R(x) \sin x$ or $\int_{-\infty}^{\infty} R(x) \cos x$ might exist.

For illustration, sps $R(x)$ has <u>simple pole</u> at $x = 0$.

Consider indented rectangle



Using analysis of previous type, we see.

$$\int_{-\infty}^{-\varepsilon} R(x)e^{ix}\,dx + \int_{\varepsilon}^{\infty} R(x)e^{ix}\,dx - \int_{C_\varepsilon} R(z)e^{iz}\,dz$$

$$= 2\pi i \sum_{\substack{Q(\alpha)=0 \\ \operatorname{Im}\alpha>0}} \operatorname{Res}_{z=\alpha} R(z)e^{iz}.$$

To compute integral on C_ε, note that near $z=0$,

$$R(z)e^{iz} = \frac{A}{z} + \underbrace{R_0(z)}_{\text{hol.}}$$

where $A = \operatorname{Res}_{z=0} R(z)e^{iz}$.

Then $\displaystyle\int_{C_\varepsilon} R(z)e^{iz}\,dz = \pi i A.$

So,

$$\lim_{\varepsilon \to 0}\left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right) R(x)e^{ix}dx = 2\pi i \left[\sum_{\substack{Q(\alpha)=0 \\ \operatorname{Im}\alpha>0}} \operatorname{Res}_{z=\alpha} R(z)e^{iz} + \frac{1}{2}\operatorname{Res}_{z=0} R(z)e^{iz}\right]$$

Def^n: The <u>principal value</u> of $\int_{-\infty}^{\infty} R(z)e^{iz}dz$ is defined as

$$\text{p.v}\int_{-\infty}^{\infty} R(\cancel{z})e^{i\cancel{z}}d\cancel{z} = \lim_{\varepsilon \to 0}\left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right) e^{ix}R(x)\,dx.$$

if $x\cancel{z} = 0$ is a simple pole.

Since $R(z)\sin z$ will have a removable sing at $z=0$, clearly

$$\lim_{\varepsilon \to 0}\int_{-\infty}^{-\varepsilon} R(x)\sin x\,dx + \int_{\varepsilon}^{\infty} R(x)\sin x\,dx.$$

$$= \int_{-\infty}^{\infty} R(x)\sin x\,dx.$$

and so

$$\int_{-\infty}^{\infty} R(x)\sin x\,dx = \operatorname{Im}\left(\text{p.v}\int_{-\infty}^{\infty} R(x)e^{ix}dx\right)$$

Example: $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx$.

By the above argument.

$$p.v \int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx = \pi i \operatorname{Res}_{z=0} e^{iz}/z = \pi i$$

So

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$$