Applications of Argument Principle

1) Rouche's Theorem

Thm: $f, g : \Omega \to \mathbb{C}$ hol, $\gamma \subset \Omega$ be a simple curve s.t $\text{Int}(\gamma) \subset \Omega$. If

$$|f(z) - g(z)| < |g(z)|$$

$\forall z \in \gamma$, then $f(z)$ and $g(z)$ have same # of zeroes (counting mult.) in $\text{Int}(\gamma)$.

Pf: First note that $g(z), f(z) \neq 0$ $\forall z \in \gamma$.

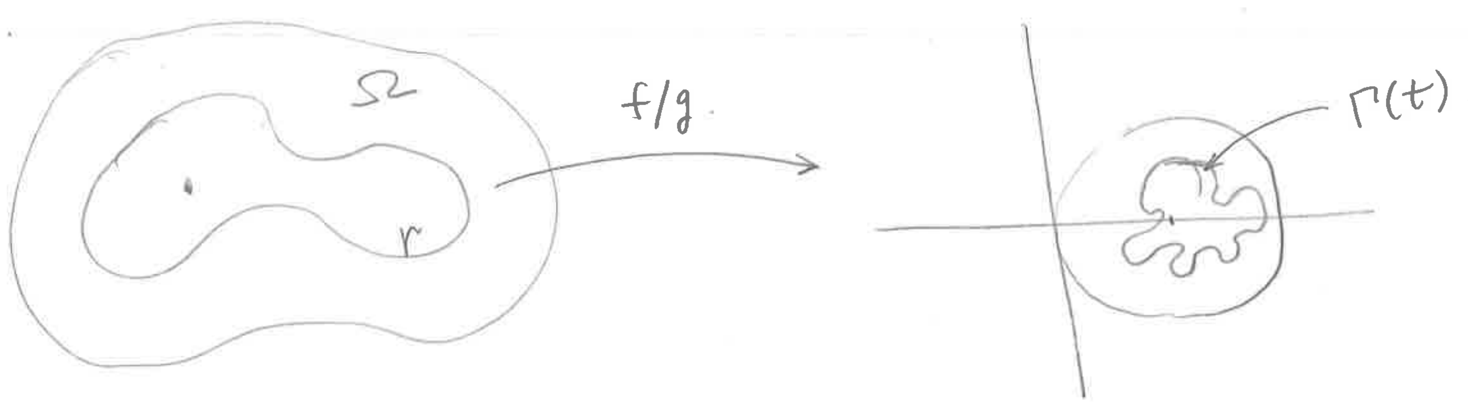
So on γ,

$$\left| \frac{f(z)}{g(z)} - 1 \right| < 1$$

Sps $\gamma : [a,b] \to \Omega$.
Define $\Gamma : [a,b] \to \mathbb{C}$ by

$$\Gamma(t) = \frac{f(\gamma(t))}{g(\gamma(t))}$$

Then $\forall t$, $|\Gamma(t) - 1| < 1$ i.e $\Gamma(t) \in D_1(1)$.



Since $0 \in \mathrm{Ext}(\Gamma)$

$$n(\Gamma, 0) = 0.$$

But $$n(\Gamma, 0) = \int_{\Gamma} \frac{dw}{w}.$$

Put $w = f(r(t))/g(r(t))$,

$$dw = \frac{g'(r(t)) \cdot r'(t) f(r(t)) - f'(r(t)) \cdot r'(t)\, g(r(t))}{g^2(r(t))} dt.$$

$$= \frac{g'(r(t)) \cdot f(r(t)) - f'(r(t))\, g(r(t))}{g^2(r(t))} \cdot r'(t)\,dt$$

$$\Rightarrow \frac{dw}{w} = \left[\frac{g'(r(t))}{g(r(t))} - \frac{f'(r(t))}{f(r(t))}\right] \cdot r'(t)\,dt.$$

$$\Rightarrow 0 = n(\Gamma, 0) = \int_a^b \frac{g'(r(t))}{g(r(t))} \cdot r'(t)\,dt \int_a^b \frac{f'(r(t))}{f(r(t))} r'(t)\,dt$$

$$\cancel{\Rightarrow} \quad = \int_r \frac{g'(z)}{g(z)} dz - \int_r \frac{f'(z)}{f(z)} dz$$

Applying argument principle, we prove the theorem

Rk. In application, we will want to find # of zeroes of $f(z)$. We will choose $g(z)$, whose zeroes are easy to find.

Example: Find # of zeroes of $z^4 - 6z + 3$ in $1 < |z| < 2$.

# of zeroes in $|z| < 2$: Let $f(z) = z^4 - 6z + 3$.
$g_1(z) = z^4$

On $|z| = 2$,

$$|f(z) - g_1(z)| = |-6z + 3| \le 6|z| + 3 = 12 + 3 = 15 < |z|^4 = |g_1(z)|.$$

Rouche's $\Rightarrow$ f & g_1 have same # of zeroes in $|z| < 2$.

But z^4 has a zero at only $z = 4$, but order is 4.

So f has 4-zeroes in $|z| < 2$.

# of zeroes in $|z| < 1$. $g_2(z) = -6z$,

On $|z| = 1$,

$$|f(z) - g_2(z)| = |z^4 + 3| \le |z|^4 + 3 \le 4 < 6|z| = |g_2(z)|$$

Rouche's $\Rightarrow$ since g_2 has only zero in $|z| < 1$, f also has only one in $|z| < 1$

$\Rightarrow$ # of zeroes in $1 < |z| < 2 = 3$.

Wolfram α: gives the roots as

$$z \approx 0.51$$

$$\left.\begin{aligned} &\approx 1.604 \\ &\approx -1.05 - 1.6i \\ &\approx -1.05 + 1.6i \end{aligned}\right\} \text{ in } 1<|z|<2.$$

2) Open Mapping Principle:

Th^m If $f : \Omega \to \mathbb{C}$ hol, then f is open i.e for any open $U \subset \Omega$, $f(U)$ is open.

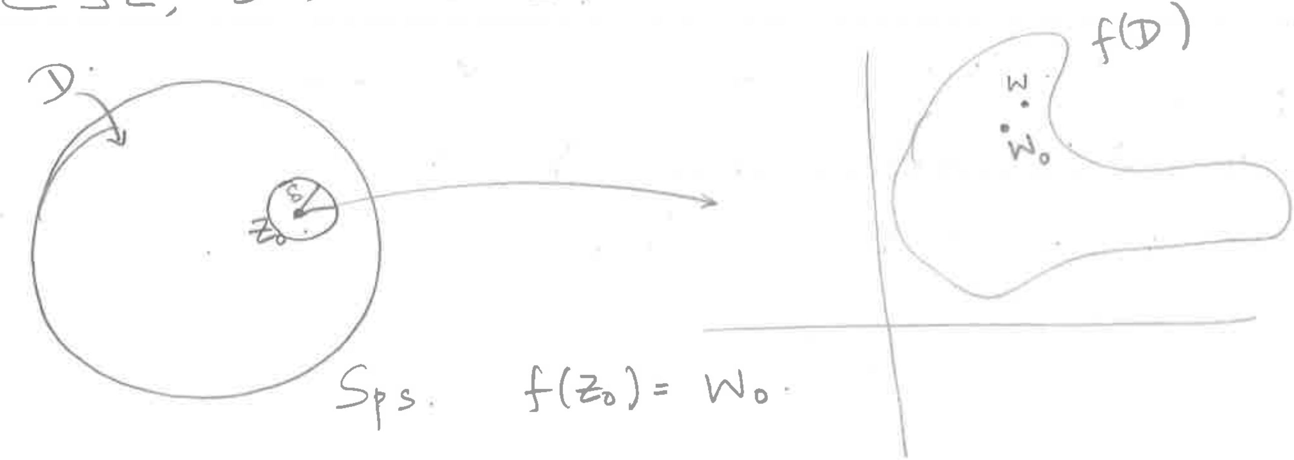
Rk: Not true for real variable functions.

For eg. consider

$$f(x) = x^2.$$

Maps $f((-1,1)) = [0,1) \leftarrow$ not open.

Pf: Enough to show $f(D)$ is open where $\overline{D} \subset \Omega$, D is a disc. Let $w_0 \in f(D)$



Sps. $f(z_0) = w_0$.

Choose ε, δ in the foll way. Since zeroes of hol functions are isolated, $\exists\ \delta > 0$ s.t $\{|z - z_0| < \delta\} \subset D$ and $f(z) \neq w_0$

$\forall\, z$ s.t. $0 < |z - z_0| \leq \delta$.

Since $|z - z_0| = \delta$ is compact, and $|f(z) - w_0|$ is not zero on this circle, $\exists\ \varepsilon$ s.t

$$|f(z) - w_0| > \varepsilon.$$

$\forall\ z$ on $|z - z_0| = \delta$.

Claim. $D_\varepsilon(w_0) \subset f(D)$.

Pf: Let $w \in D_\varepsilon(w_0)$; consider $F(z) = f(z) - w$.

$G(z) = f(z) - w_0$.

Then on $|z - z_0| = \delta$,

$$|F(z) - G(z)| = |w - w_0| \underset{\substack{\text{since} \\ w \in D_\varepsilon(w_0)}}{<} \varepsilon < |f(z) - w_0| = |G(z)|.$$

$\Rightarrow$ F & G have same roots in $|z - z_0| < \delta$.

But G has one, namely z_0.

Hence $\exists\ z$ s.t $f(z) = w$ i.e $w \in f(D)$

$\Rightarrow$ $D_\varepsilon(w_0) \subset f(D)$.

3) Inverse function theorem

Th^m: Let $f : \Omega \to \mathbb{C}$ hol. If $f'(z_0) \neq 0$, then $\exists$ $\delta > 0$ s.t $f|_{D_\delta(z_0)} : D_\delta(z_0) \longrightarrow f(D_\delta(z_0))$

is bijective, and the inverse function is also hol

Pf: Let $f(z_0) = w_0$.

Claim: $\exists$ $\delta_0, \varepsilon_0 > 0$ s.t. $\forall$ $w \in D_{\varepsilon_0}(w_0)$, $\exists$ unique $z \in D_{\delta_0}(z_0)$ s.t

$$f(z) = w.$$

Pf:

Choose $\delta_0 > 0$ s.t $f'(z) \neq 0$ $\forall$ $z \in D_{\delta_0}(z_0)$.
$f(z) \neq w_0$ $\forall$ $z \in D_{\delta_0}(z_0)$.

$\varepsilon_0 > 0$ s.t $|f(z) - w_0| > \varepsilon_0$ $\forall$ $|z - z_0| = \delta_0$.

From the proof of open-mapping theorem, $\forall$ $w \in D_{\varepsilon_0}(w_0)$, $f(z) - w = 0$ & $f(z) - w_0 = 0$ have same # of sol^n.
But $f(z) = w_0$ has only one sol^n & since $f'(z_0) \neq 0$, multiplicity of zero $= 1$.

$\Rightarrow \quad f(z) = w$ has a unique sol^n in $D_{\delta_0}(z_0)$.

Back to proof: Let $\delta < \delta_0$ s.t.

$$|f(z) - w_0| < \varepsilon_0$$

$\forall\ z \in D_\delta(z_0)$. Then $f(D_\delta(z_0)) \subset D_{\varepsilon_0}(w_0)$.

For $w \in f(D_\delta(z_0))$ there is already a $z \in D_\delta(z_0)$

s.t. $\qquad f(z) = w$.

Since $w \in D_{\varepsilon_0}(w_0)$, $z \in D_\delta(z_0) \subset D_{\delta_0}(z_0)$.

by Claim this solution is unique.

So $\quad f : D_\delta(z_0) \longrightarrow f(D_\delta(z_0))$

is injective.

By def^n it is also surjective

Done!

Rk: More generally, we have the following beautiful fact: If $f(z_0) = w_0$, $f : \Omega \to \mathbb{C}$ hol. Sps $f^{(k)}(z_0) = 0$ for $k = 0, 1, \dots m-1$.

Then $\exists\ \delta_0, \varepsilon_0 > 0$ s.t $\forall\ w \in D_{\varepsilon_0}(w_0)$ there exist 'm' distinct solutions to

$$f(z) = w$$

with $z \in D_\delta(z_0)$.

4) Max - Modulus principle

Th^m: If $f : \Omega \to \mathbb{C}$ hol. Then $|f|$ cannot attain a maximum in Ω.

Pf: Argue by contradiction

Sps $|f(z_0)| = \max\limits_{z \in \Omega} |f(z)|$. $(*)$

Let $\delta > 0$ s.t $D_\delta(z_0) \subset \Omega$.

Open mapping $\Longrightarrow$ $\exists$ $\varepsilon > 0$ s.t. $D_\varepsilon(w_0) \subset f(\Omega)$

Clearly in $D_\varepsilon(w_0)$ $\exists$ w s.t $|w| > |w_0|$.

$\Rightarrow$ $\exists$ $z \in \Omega$ s.t. $|f(z)| > |f(z_0)|$.

contradicting $(*)$.

Cor: $f : \Omega \to \mathbb{C}$ hol, $\overline{\Omega}$ compact s.t f extends to a cont. function on $\overline{\Omega}$.

Then

$$\sup_{\Omega} |f(z)| \leq \sup_{\partial\Omega} |f(z)|.$$

<u>Rk</u>: Min. modulus principle <u>cannot</u> be true since f might have zeroes in Ω.
Clearly if $f(z) \neq 0 \ \forall z \in \Omega$, then $|f|$ cannot have an interior minimum.