Residue Theorem & Argument Principle

- Residue Theorem.

Def^n: Let $\Omega \subseteq \mathbb{C}$ open, $p \in \Omega$ and $f : \Omega_p^* \to \mathbb{C}$ hol. If

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-p)^n$$

on some disc $D \subset \Omega$, we define the residue of f at $z = p$ by

$$\boxed{\operatorname{Res}_p f(z) = a_{-1}}$$

Rk 1: We have already seen that

$$a_{-1} = \frac{1}{2\pi i} \int_{\partial D} f(z)\, dz$$

i.e

$$\int_{\partial D} f(z)\, dz = 2\pi i \operatorname{Res}_p f(z)$$

Th^m (Residue Th^m): Let $\Omega \subset \mathbb{C}$ open and $f : \Omega \setminus \{p_1, \dots\} \to \mathbb{C}$ hol. $\Gamma \subset \Omega$ closed curve. Assume

1) $p_1, \dots$ are isolated

2) No p_j lies on Γ

3) Γ is simple & $\operatorname{Int}(\Gamma) \subset \Omega$

Then

$$\int_{\Gamma} f(z)\, dz = 2\pi i \sum_{p_j \in \operatorname{Int}(\Gamma)} \operatorname{Res}_{p_j} f(z)$$

if γ is given the positive orientation:

Rk 2. 1) If Ω is simply connected, $\mathrm{Int}(\gamma) \subset \Omega$ is automatic.

2) More generally, if γ is not simple, we can prove

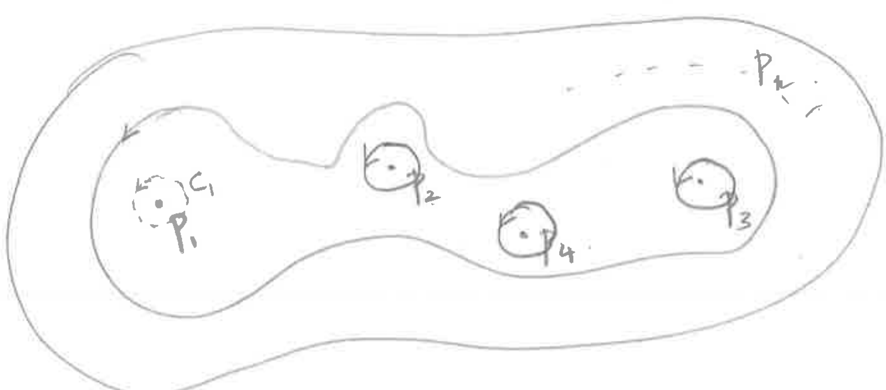
$$\int_\gamma f(z)\,dz = 2\pi i \sum_{p \in \mathrm{Sing}(f)} n(\gamma, p) \cdot \mathrm{Res}_p f(z).$$

where $\mathrm{Sing}(f) =$ set of singularities of f
(i.e $p_1, p_2, \ldots$)

$n(\gamma, p) =$ index of γ around p.

Since γ is compact and p_j do not have a limit point in Ω, $\exists$ only finitely many in $\mathrm{Int}(\gamma)$. If $p \in \mathrm{Sing}(f) \cap \mathrm{Ext}(\gamma)$, $n(\gamma, p) = 0$.
So the sum on the right is a finite sum.

Pf:



Arrange the singularities s.t $p_1, \ldots p_N \in \mathrm{Int}(\gamma)$
$C_j =$ circle centered at p_j with +ve orientation
s.t $p_k \notin \mathrm{Int}(C_j)$ if $k \neq j$. & $C_j \cap C_k = \emptyset$.

Generalized Cauchy $\Longrightarrow$

$$\int_{\Gamma} f(z)\,dz = \int_{C_1} f(z)\,dz + \cdots + \int_{C_N} f(z)\,dz$$

By Rk 1,

$$\int_{C_j} f(z)\,dz = 2\pi i \operatorname{Res}_{p_j} f(z)$$

$\forall \; j = 1, \dots, N$

Summing up we prove Th^m

Example: Find $\displaystyle\int_{|z|=1} \frac{\sin(1/z)}{z-3}\,dz$

Check: Laurent Series at $z=0$ is

$$-\frac{1}{3}\left(\frac{1}{z} - \frac{1}{6z^3} + \frac{1}{120 z^5} \cdots\right)\left(1 + \frac{z}{3} + \frac{z^2}{9} + \cdots\right)$$

So coeff of $1/z$

$$a_{-1} = -\sum_{k=0}^{\infty} \frac{(-1)^k}{3^{2k+1}(2k+1)!} = -\sin\left(\frac{1}{3}\right)$$

No other sing in $|z|<1$. So

$$\int_{|z|=1} \frac{\sin(1/z)}{z-3} = -2\pi i \sin\left(\frac{1}{3}\right)$$

• Residue at infinity: Let f hol on $\mathbb{C}$ with finitely many isolated sing. $p_1, p_2, \dots, p_N$. Then $z=\infty$ is an isolated sing. If $A_R = \{|z|>R\}$ is a neighborhood s.t $p_1, \dots, p_N \in \{|z|<R\}$

Then we define

$$\mathrm{Res}_\infty f(z) := -\frac{1}{2\pi i} \int_{C_R} f(z)\, dz .$$

$C_R \leftarrow$ circle $|z| = R$.

The $-$ve sign since if we consider C_R a boundary of A_R, to keep A_R on left we have to traverse in clock-wise direction which is opp of the usual orientation of C_R.

Rk From definition & residue theorem

$$\sum_{j=1}^{N} \mathrm{Res}_{p_j} f(z) + \mathrm{Res}_\infty f(z) = 0 .$$

Th^m: With notation as

-ve sign

$$\mathrm{Res}_\infty f(z) = \mathrm{Res}_0 \frac{1}{z^2} f\left(\frac{1}{z}\right)$$

Pf: let $z = 1/w$, then $dz = -dw/w^2$.

$$\mathrm{Res}_\infty f(z) = -\frac{1}{2\pi i} \int_{|z|=R} f(z)\, dz = \frac{1}{2\pi i} \int_{|w|=1/R} f\left(\frac{1}{w}\right) \cdot \frac{dw}{w^2}$$

$$= \mathrm{Res}_0 \frac{1}{w^2} f\left(\frac{1}{w}\right)$$

Example $f(z) = \sin(1/z)/(z-3)$

$$\mathrm{Res}_\infty f(z) = \mathrm{Res}_0 \frac{1}{z^2} \frac{\sin(z)}{1/z - 3} = \mathrm{Res}_0 \left(\frac{\sin z}{z}\right)\left(\frac{1}{1-3z}\right)$$

$= 0$ since both are hol at $z = 0$

Also easy to see $\mathrm{Res}_3 f(z) = \sin(1/3)$.

So by Rk above

$$\mathrm{Res}_0 f(z) = -\sin(1/3).$$

$$\Rightarrow \quad \operatorname{Res}_{z=0} f(z) = -\sin\left(\frac{1}{3}\right).$$

• Residue at a pole

Th^m: Let f have a pole of order m at p.

Then,

$$\operatorname{Res}_{z=p} f(z) = \frac{1}{(m-1)!} \left.\frac{d^{m-1}}{dz^{m-1}}\right|_{z=p} (z-p)^m f(z).$$

Pf: Near p,

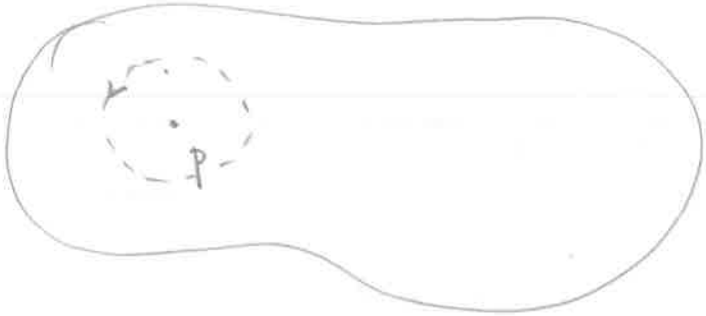
$$f(z) = \frac{a_{-m}}{(z-p)^m} + \frac{a_{-m+1}}{(z-p)^{m-1}} + \cdots + \frac{a_{-1}}{(z-p)} + a_0 + \cdots$$

$$\Rightarrow \quad (z-p)^m f(z) = a_{-m} + a_{-m+1}(z-p) + \cdots + a_{-1}(z-p)^{m-1} + a_0(z-p)^m + \cdots$$

$$\Rightarrow \quad \frac{d^{m-1}}{dz^{m-1}} (z-p)^m f(z) = (m-1)!\, a_{-1} + a_0\, m!\,(z-p) + \cdots$$

Plugging in $z=p$ we obtain the result.

• <u>Argument Principle</u>. Sps $f : \Omega \to \mathbb{C}$ hol., and $p \in \Omega$ is a <u>zero of order 'm'</u>.



On a small disc, $D_\varepsilon(p)$ s.t $\overline{D_\varepsilon(p)} \subset \Omega$.

$$f(z) = (z-p)^m \cdot g(z)$$

where $g(z) \neq 0$ on $\overline{D_\varepsilon(p)}$.

Then $$f'(z) = m(z-p)^{m-1} g(z) + (z-p)^m g'(z).$$

$$\Rightarrow \quad \frac{f'(z)}{f(z)} = \frac{m}{(z-p)} + \frac{g'(z)}{g(z)}$$ ← hol on $D_\varepsilon(p)$.

$\Rightarrow$ By Cauchy's theorem

$$\boxed{\frac{1}{2\pi i}\int_{|z-p|=\varepsilon} \frac{f'(z)}{f(z)}\, dz = m}$$

Similarly if $p \in \Omega$ is a <u>pole</u> of order 'm'.
Then near p. on $D_\varepsilon(p)$

$$f(z) = \frac{g(z)}{(z-p)^m}, \quad g(z) \neq 0 \text{ on } \overline{D_\varepsilon(p)}$$

$$\Rightarrow \quad f'(z) = -\frac{m\, g(z)}{(z-p)^{m+1}} + \frac{g'(z)}{(z-p)^m}.$$

So $$\frac{f'(z)}{f(z)} = -\frac{m}{(z-p)} + \frac{g'(z)}{g(z)} \leftarrow \text{hol.}$$

Again by Cauchy.

$$\boxed{\frac{1}{2\pi i}\int_{|z-p|=\varepsilon} \frac{f'(z)}{f(z)}\, dz = -m}.$$

Generalizing this we have.

<u>Th</u>

$\underline{\text{Th}^m}$ (Argument Principle). Let Ω simply connected, $f : \Omega \to \hat{\mathbb{C}}$ meromorphic. If γ a closed, simple + ve oriented curve s.t. γ does not contain any zero or pole of f. Then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\, dz = \#(\text{zeroes in } \mathrm{Int}(\gamma)) - \#(\text{poles in } \mathrm{Int}(\gamma))$$

counted with multiplicities.

$\underline{\text{Pf}}$: Use generalized Cauchy to reduce to integrals on circles centered at zeroes & poles (similar to proof of residue theorem). For circles we have proved the theorem in remarks preceding the Theorem.

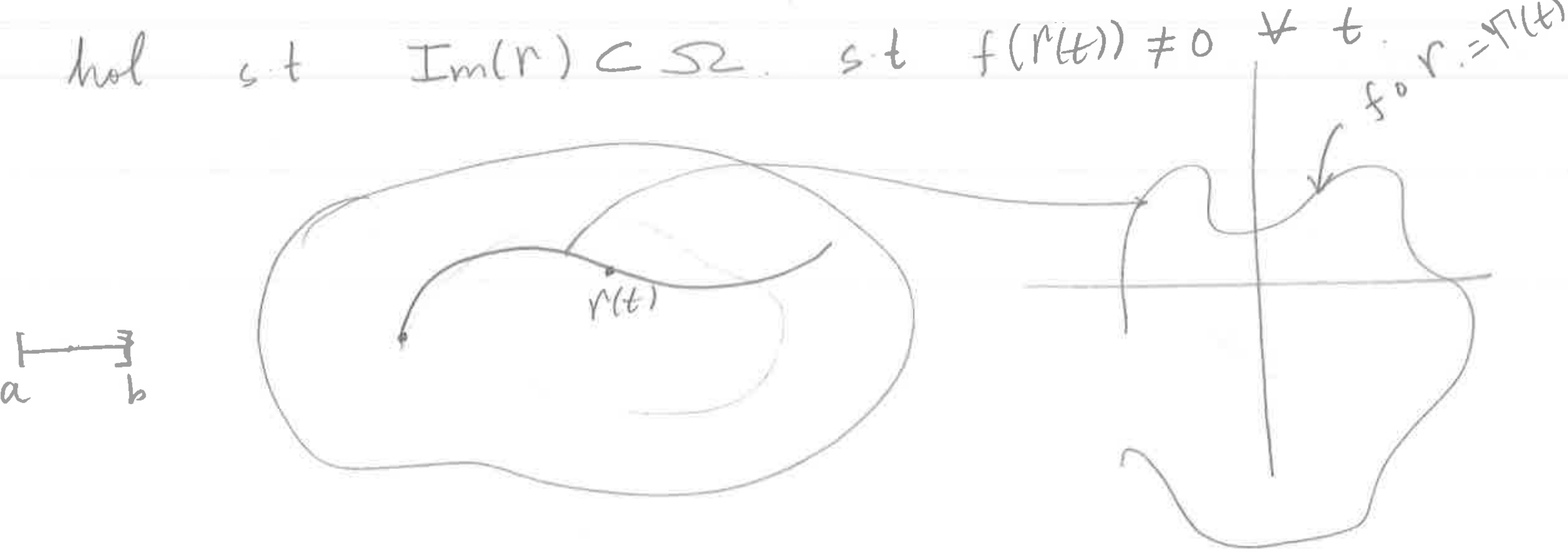
- <u>Argument principle as Index</u>.

Recall, the index

$$n(\gamma, 0) = \frac{1}{2\pi i}\int_{\gamma} \frac{dw}{w}$$

calculates the change in arg w as one traverses around γ.

Let $\gamma : [a, b] \to \mathbb{C}$ be a curve and $f : \Omega \to \mathbb{C}$ hol s.t $\operatorname{Im}(\gamma) \subset \Omega$. s.t $f(\gamma(t)) \neq 0 \ \forall t$



Then $\Gamma(t) = f(\gamma(t))$ is another curve

$\Gamma'(t) = f'(\gamma(t))\,\gamma'(t)$.

$$n(\Gamma, 0) = \frac{1}{2\pi i}\int_{\Gamma} \frac{dw}{w}$$, put $w = f(z)$; $z \in \gamma$.
$dw = f'(z)\,dz$.

$$\frac{1}{2\pi i}\int_{\gamma} \frac{f'(z)}{f(z)}\,dz \leftarrow$$ measures the change in arg $f(z)$ as z moves along γ.