LECTURE-1 : INTRODUCTION TO COMPLEX ANALYSIS

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1. Introduction

Complex analysis is one of the most beautiful branches of mathematics, and one that lies at the heart of several other subjects, such as topology, algebraic geometry, Fourier analysis, and number theory.

The main objects in calculus are real valued functions defined on intervals. The starting point in complex analysis is to extend the notion of functions to include *complex valued* functions

$$f : \Omega \to \mathbb{C}$$

defined on subsets $\Omega \subset \mathbb{C}$ of complex numbers. Recall that complex numbers can be added, subtracted, multiplied and divided (if non-zero) just like real numbers. Every complex number can be written in the form

$$z = x + iy$$

where x and y are real numbers. So complex numbers can be identified as a set with Euclidean plane $\mathbb{R}^2$. The addition of complex numbers is also equivalent to addition of vectors in $\mathbb{R}^2$. So it might appear as if we are not adding much, and that nothing is lost by simply treating the complex valued function as a two variable vector field. In fact this is true, as we will see later, when talking about limits and continuity.

But there is one key difference between $\mathbb{R}^2$ and $\mathbb{C}$, that of multiplication and division. Indeed things change rapidly when we restrict our attention to *holomorphic* functions, or functions for which

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists and is finite. The important point being that h could be a complex number. Formally this definition is identical to that of a differentiable function in one-variable calculus. But quite surprisingly the mere change of perspective, the fact that h is allowed to take complex values as it goes to zero, produces beautiful new phenomenon that have no counterparts in one-variable calculus, or indeed even multivariable calculus. We now summarize some of these remarkable consequences of holomorphicity.

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- **Power Series expansion.** As we remarked above, complex valued functions can be thought as mapping between sets in $\mathbb{R}^2$. We will prove later in the course that for a holomorphic function, partial derivatives of all orders exist. And moreover, one also has that the Taylor series at ever point converges to the function value. Recall that this is not true for one-variable functions. For instance if
$$f(x) = \begin{cases} e^{-1/x^2}, \ x > 0 \\ 0, \ x \leq 0, \end{cases}$$
then it is easy to see that at $x = 0$, derivative of any order is zero. So the Taylor series of the function at $x = 0$ is zero, but the function is clearly not zero.
- **Analytic continuation.** If two holomorphic functions are defined on an open connected domain are equal in a small open neighbourhood of a point, no matter how small the neighbourhood is, have to be equal throughout the domain of their definition.
- **Good convergence properties.** If a sequence of holomorphic functions converges uniformly, the limit function is again holomorphic. This is not true for differentiable one-variable functions. For instance, if $f_n : [-1, 1] \to \mathbb{R}$ is defined by
$$f_n(x) = \sqrt{\frac{1}{n} + x^2},$$
then one can show that $f_n \to |x|$ uniformly, but $|x|$ is not differentiable.
- **Liouville property.** A bounded holomorphic function defined on all of $\mathbb{C}$ is forced to be a constant.

Part of the richness of the theory of holomorphic functions comes from the variety in the methods used to study the subject. We next summarize the approaches that we will touch upon in this course.

- **Partial differential equations.** It turns out that real and imaginary parts of holomorphic functions, thought of as real valued two-variable functions, satisfy a system of first-order partial differential functions, called the *Cauchy-Riemann* equations. As a consequence of this, the real and imaginary parts are *harmonic functions*. The theory of harmonic functions is rather well developed, and could be potentially exploited to study holomorphic functions. We will only touch upon the Cauchy-Riemann equations, but will not pursue this approach further. We will instead focus on integral methods.
- **Integral methods.** The viewpoint that we will adopt is centered on a remarkable formula called the Cauchy's integral formula. We will develop a notion of integration of complex valued functions along curves, a generalization of the notion of line integrals in multi-variable calculus. The fundamental fact, which will be the theoretical basis for the rest of the course, is that the complex integral of

a function around a closed curve is zero if it is holomorphic on the interior of the curve. If the real and imaginary parts of the holomorphic function are assumed to have continuous partial derivatives, this result follows from Green's theorem. We will give an independent proof, not because we wish to be smart, but because remarkably this theorem will *imply* that the real and imaginary parts of the holomorphic function indeed have not only continuous partial derivatives but have partial derivatives of all orders.

- **Power series methods.** As remarked above, every holomorphic function is represented by a power series. Since power series are algebraic objects, for the most part they can also be manipulated as if they were polynomials. Thus algebraic methods can be used to study holomorphic functions.
- **Geometric mathods.** An elementary but beautiful fact is that holomorphic functions, thought of as mappings (or transformations) between sets in $\mathbb{R}^2$ are *conformal maps*. That is, holomorphic mappings preserve angles between curves, and stretch the distances. We will study some standard examples of conformal maps. We will end the course with the following deep fact, first discovered by Riemann - Any domain in the complex plain which does not have a 'hole' and which is not the entire complex plane, can be mapped conformally to a disc centered at the origin of radius one. In the course, we will discuss a proof due to Koebe.

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