LECTURE-12 : HOMOTOPIES AND SIMPLE CONNECTEDNESS

VED V. DATAR*

In the previous lecture, we saw that the line integral of holomorphic function along any closed path in a disc is zero. The aim of this lecture is to study continuous deformations of curves, and to present a first generalization of Cauchy's theorem, namely to simply connected domains.

Continuous deformations and Simple connectedness

Two curves γ_0 and γ_1 are said to be *homotopic* to each other, in a domain Ω, if roughly speaking, one can be continuously deformed into the other without leaving Ω. More precisely, let $\gamma_0, \gamma_1 : [a, b] \to \Omega$ be two continuous parametric curves in Ω with common end points, that is

$$\gamma_0(a) = \gamma_1(a) = p,\ \gamma_0(b) = \gamma_1(b) = q.$$

Then they are said to be *homotopic* in Ω, denoted by $\gamma_0 \sim_\Omega \gamma_1$, if there exists a continuous function $\Gamma : [0, 1] \times [a, b] \to \Omega$ such that

- $\Gamma(0, t) = \gamma_0(t)$ and $\Gamma(1, t) = \gamma_1(t)$.
- $\Gamma(s, 0) = p$ and $\Gamma(s, b) = q$.
- For each $s \in [0, 1]$, $\gamma_s(t) := \Gamma(s, t)$ is a piecewise smooth curve. A closed curve is said to be homotopic to a point, if it is homotopic to a constant curve.

CAUTION: In the lecture on the homotopy theorem, for two curves to be homotopic, it was not required for them to have common initial and end points. But as clarified in the subsequent lecture, let us choose to adopt the convention that homotopic curves do have common initial and end points.

Example 0.1. *The semicircles $\gamma_0, \gamma_1 : [0, \pi] \to \mathbb{C}$ given by*

$$\gamma_0(t) = e^{it},\ \gamma_1(t) = e^{-it}$$

are homotopic in $\mathbb{C}$. A precise homotopy can be given by linear interpolation, that is

$$\Gamma(s, t) = se^{-it} + (1 - s)e^{it}.$$

This is not a homotopy over the punctured plane $\mathbb{C}^*$ since at $s = 1/2$ and $t = \pi/2$ it crosses the origin. In fact it is easy to see, for instance by the intermediate value theorem from calculus, that any homotopy between the two semi-circles has to pass through the origin at some point.

Date: 24 August 2016.

Example 0.2. *The circle* $\gamma(t) = e^{it}$ *is homotopic to the point* $(1, 0)$ *in* $\mathbb{C}$. *One such homotopy is given by*

$$\Gamma(s, t) = s + (1 - s)e^{it}.$$

Geometrically this is a sequence of shirking circles with the center approaching $z = 1$. *More generally, any curve* $\gamma : [a, b] \to \mathbb{C}$ *is homotopic (in* $\mathbb{C}$*) to a point via the homotopy*

$$\Gamma(s, t) = s\gamma(a) + (1 - s)\gamma(t).$$

The basic theorem that we want to prove is that line integrals of holomorphic functions are invariant under continuous deformations. More precisely we have the following.

Theorem 0.1. *If* f *is holomorphic in* Ω, *then*

$$\int_{\gamma_0} f(z)\, dz = \int_{\gamma_1} f(z)\, dz$$

whenever γ_0 *and* γ_1 *are homotopic curves in* Ω.

Proof. Let $\Gamma : [0, 1] \times [a, b] \to \Omega$ be a homotopy connecting γ_0 to γ_1, and let K be the image of Γ in Ω. Then K is compact.

Claim 1. *There exists a small* $\varepsilon > 0$ *such that the ball* $B_{3\varepsilon}(z) \subset \Omega$ *for any* $z \in K$.

If not, then there would be a sequence of points in $z_n \in K$ and $w_n \in \partial\Omega$ such that $|z_n - w_n| < 1/n$. But by compactness there is a convergent subsequence with the limit point in K. On the other hand, the sequence must converge to the boundary of Ω, a contradiction. Let us fix such a $\varepsilon > 0$ once and for all.

Next, by uniform continuity (recall that continuous functions on compact sets are uniformly continuous), there exists a $\delta > 0$ such that

$$|s_1 - s_2| < \delta \implies \sup_{[a,b]} |\Gamma(s_1, t) - \Gamma(s_2, t)| < \varepsilon.$$

Now fix s_1, s_2 satisfying $|s_1 - s_2| < \delta$, and choose discs $D_0, D_1, \cdots, D_n$ of radius 2ε, and points $\{z_0, \cdots, z_{n+1}\}$ and $\{w_0, \cdots, w_{n+1}\}$ lying on curves γ_{s_1} and γ_{s_2} respectively with $z_0 = w_0$ the initial point and $z_{n+1} = w_{n+1}$ the final point, and such that

$$\gamma_{s_1}, \gamma_{s_2} \in \cup_{j=0}^{n} D_j$$
$$z_j, z_{j+1}, w_j, w_{j+1} \in D_j$$

Denote the part of the curve γ_{s_1} lying between z_j and z_{j+1} be $C_j^{(1)}$, and let the corresponding part on γ_{s_2} between w_j and w_{j+1} be $C_j^{(2)}$. Furthermore, for $j = 0, 1, \cdots, n$ let γ_j be the *closed* curve that consists

- The part of γ_{s_2} going from w_j to w_{j+1}
- the straight line going from w_{j+1} to z_{j+1}

- The part of $-\gamma_{s_1}$ going from z_{j+1} to z_j and
- the straight line from z_j to w_j.

When $j = 0$ or n the curve actually consists of only three parts since $z_0 = w_0$ and $z_{n+1} = w_{n+1}$. In any case, γ_j is a piecewise smooth path that completely lies in D_j. So by Cauchy's theorem for discs,

$$\int_{\gamma_j} f(z)\, dz = 0.$$

Moreover, the common parts of the curves γ_j and γ_{j+1} (namely the straight lines) have exactly opposite orientations. Hence we can write $\gamma_{s_2} - \gamma_{s_1} = \sum_{j=0}^{n} \gamma_j$, and so

$$\int_{\gamma_{s_2}} f(z)\, dz - \int_{\gamma_{s_1}} f(z)\, dz = \sum_{j=0}^{n} \int_{\gamma_j} f(z)\, dz = 0.$$

So we have proved that for any s_1 and s_2 such that $|s_1 - s_2| < \delta$,

$$\int_{\gamma_{s_2}} f(z)\, dz = \int_{\gamma_{s_1}} f(z)\, dz.$$

But then we can reach from $s = 0$ to $s = 1$ in finitely many steps of size less than δ, and applying the argument finitely many times we obtain that

$$\int_{\gamma_0} f(z)\, dz - \int_{\gamma_1} f(z)\, dz.$$

□

A domain is called *simply connected* if any two curves with the same end points are homotopic. Or equivalently, any closed curve is homotopic to a point (which is to say, it homotopic to a constant curve). Then as a consequence of the above theorem, we have the following.

Corollary 0.1. *Any holomorphic function in a simply connected domain has a primitive. As a consequence, if Ω is simply connected, and $f : \Omega \to \mathbb{C}$ holomorphic, then*

$$\int_{\gamma} f(z)\, dz = 0$$

for all closed curves $\gamma \subset \Omega$.

Proof. It is enough to prove the existence of a primitive. The proof is along the lines of the proof for existence of primitives on disc that was used in the proof of Cauchy's theorem. So we fix a $p \in \Omega$, and define

$$F(z) = \int_{\gamma_z} f(w)\, dw,$$

where the integral is along some path γ_z joining p to z. If we choose another path $\tilde{\gamma}$ joining the two points, then since the domain is simply connected, γ and $\tilde{\gamma}$ will be homotopic, and then by Theorem 0.1 the integral would be the same. Hence our definition is actually independent of the path. By openness of Ω, for any h small, the straight line joining z to $z+h$ will lie entirely in Ω, and we call this path as l. Then $\gamma_{z+h} - l$ and γ_z are both piecewise smooth paths joining p to z, so once again by simple connectedness of Ω and Theorem 0.1

$$\int_{\gamma_{z+h}} f(w)\,dw - \int_l f(w)\,dw = \int_{\gamma_z} f(w)\,dw,$$

or equivalently

$$F(z+h) - F(z) = \int_l f(w)\,dw.$$

Then the same argument as before implies that $F(z)$ is holomorphic with

$$F'(z) = f(z).$$

□

Example 0.3. $\mathbb{C}$ *is simply connected. More generally, any disc is simply connected. In fact, any curve in a disc* $D_p(R)$ *can be shrunk to the center point. To be more precise, if* $\gamma : [a,b] \to D_p(R)$ *is a curve, then consider the homotopy* $\Gamma : [0,1] \times [a,b] \to D_p(R)$ *given by*

$$\Gamma(s,t) = a + (1-s)(\gamma(t) - a) = sa + (1-s)\gamma(t).$$

So Cauchy's theorem is only a special case of the above theorem.

Jordan curve theorem and simple connectivity

The homotopy property is not easy to check, and hence does not provide a practical way to identify simply connected domains. In this section we will see that loosely speaking, simply connected domains are domains without a hole. Recall that a *simple, closed* curve is a curve $\gamma : [a,b] \to \Omega$ such that γ is injective on $[a,b)$ and $\gamma(a) = \gamma(b)$. Such a curve is called a *Jordan curve*. We then have the following deep theorem, which we state without proof.

Theorem 0.2 (Jordan curve theorem)**.** *Let* γ *be a Jordan curve and* C *be it's image. Then it's complement* $\mathbb{C}\backslash C$ *consists of exactly two open connected subsets. One of these components is bounded while the other is unbounded.*

The bounded component is called the *interior* and the unbounded component is called the *exterior*. While intuitively obvious, the proof is extremely non-trivial. So much so that the the theorem is notorious for numerous incorrect proofs from well known mathematicians. In fact it will not be an exaggeration to say that attempts to prove this theorem led to the modern development of algebraic topology. We then have the following characterization of simply connected domains.

Theorem 0.3. Ω *is simply connected if and only if for every Jordan curve in* Ω*, the interior is contained in* Ω*.*

In the lecture on the index of a curve, we will prove a much more general version of this, and hence we defer the proof for later. The equivalent characterization makes it easier to test domains for simple connectivity.

Example 0.4. $\mathbb{C}^*$ *is not simply connected. Any circle* $|z| = R$ *is an example of a Jordan curve whose interior is not contained in the set.*

Example 0.5. $\Omega = \mathbb{C} \setminus \{z \mid z \in \mathbb{R},\ z \leq 0\}$ *is simply connected. More generally, the complement of any infinite ray is also simply connected. To see this, note that an infinite ray from a point interior to any closed simple curve has to intersect the curve at least once. So if* Ω *was not simply connected there would be a simple closed curve* $\gamma \subset \Omega$ *whose interior is not completely contained in* Ω*. So there would be a point* $z \in Int(\gamma)$ *such that* $z \in \mathbb{R}$ *and* $z \leq 0$*. But then the infinite ray going from* z *to* $(-\infty, 0)$ *will have to intersect* γ *somewhere contradicting the fact that* $\gamma \subset \Omega$*. Hence* Ω *has to be simply connected.*

Generally speaking, it is far easier to prove that a domain is NOT simply connected, since we only have to exhibit one curve whose interior is not completely contained in the domain.

$*$ Department of Mathematics, UC Berkeley
E-mail address: vvdatar@berkeley.edu