Lecture :

• Last time : 1) Fractional linear transform.

$$Tz = \frac{az+b}{cz+d}, \quad ad-bc.$$

2) $\exists$ unique T, s.t for distinct points $p, q, r \in \hat{\mathbb{C}}$

~~as~~ $T(p) = 1, \ T(q) = 0, \ T(r) = \infty$.

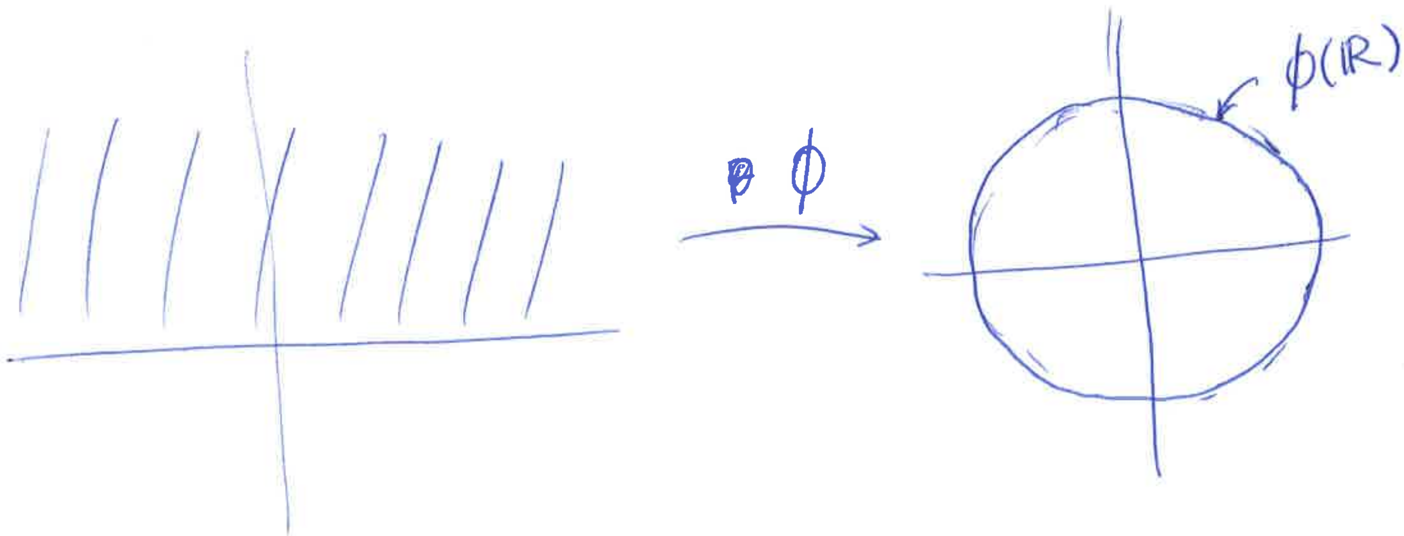
3) T takes gen. circle to gen. circle.

• Upper Half plane : $\mathbb{H} = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}$

$$\mathbb{D} = \{ z \in \mathbb{C} \mid |z| < 1 \}.$$

Then $\mathbb{H}$ and $\mathbb{D}$ are conformally equivalent.

(Cayley map): F.L.T mapping $\mathbb{R}$ to $\mathbb{D}$.



$$\phi(z) = \frac{z-i}{z+i}$$

$$|\phi(z)|^2 = \left(\frac{z-i}{z+i}\right)\left(\frac{\bar{z}+i}{\bar{z}-i}\right) = \frac{|z|^2 + i(z-\bar{z}) + 1}{|z|^2 - i(z-\bar{z}) + 1}.$$

But $\operatorname{Im} z = \dfrac{z-\bar{z}}{2i} \Rightarrow 2\operatorname{Im} z = -i(z-\bar{z})$

$$\Rightarrow |\phi(z)|^2 = \frac{|z|^2 + 1 - 2\operatorname{Im} z}{|z|^2 + 1 + 2\operatorname{Im} z} < 0 \quad \text{since } \operatorname{Im} z > 0.$$

So $\phi : \mathbb{H} \to \mathbb{D}$ and <u>hol</u> since only pole is $z = -i \notin \mathbb{H}$.

Since F.L.T it is <u>conformal</u>.

It is also <u>surjective</u>. To see this consider $w \in \mathbb{D}$ and solve

$$w = \frac{z-i}{z+i}$$

$$\Rightarrow \quad z = i\frac{(1+w)}{1-w}.$$

$$\operatorname{Im} z = \operatorname{Re}\left(\frac{1+w}{1-w}\right) = \frac{1}{2}\left[\frac{1+w}{1-w} + \frac{1+\bar{w}}{1-\bar{w}}\right]$$

$$= \frac{1-|w|^2}{|1-w|^2} > 0 \quad \text{if } |w| < 1.$$

<u>Prop</u>: $\mathbb{H}$ and $\mathbb{D}$ are conformally equivalent

Note: $\phi(\infty) = 1$, $\phi(0) = -1$; $\phi(1) = \dfrac{1-i}{1+i} = -i$. $|\cancel{\phi(x)}|$

$= \dfrac{(1-i)^2}{2} = \cancel{0} - i$

$1 - i = -i - i^2$

OTHER EXAMPLES.

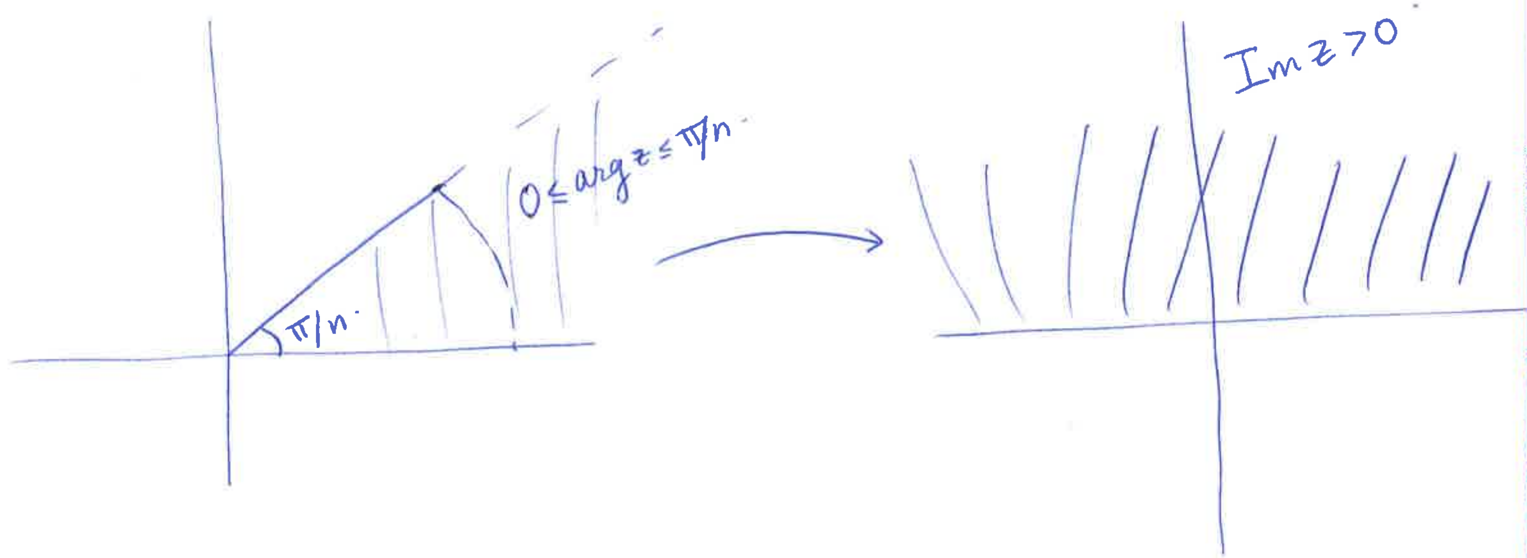
1) Rotation and dilation

Rotate by angle 'θ' $\longrightarrow$ ~~and~~ $T(z) = e^{i\theta} z$.

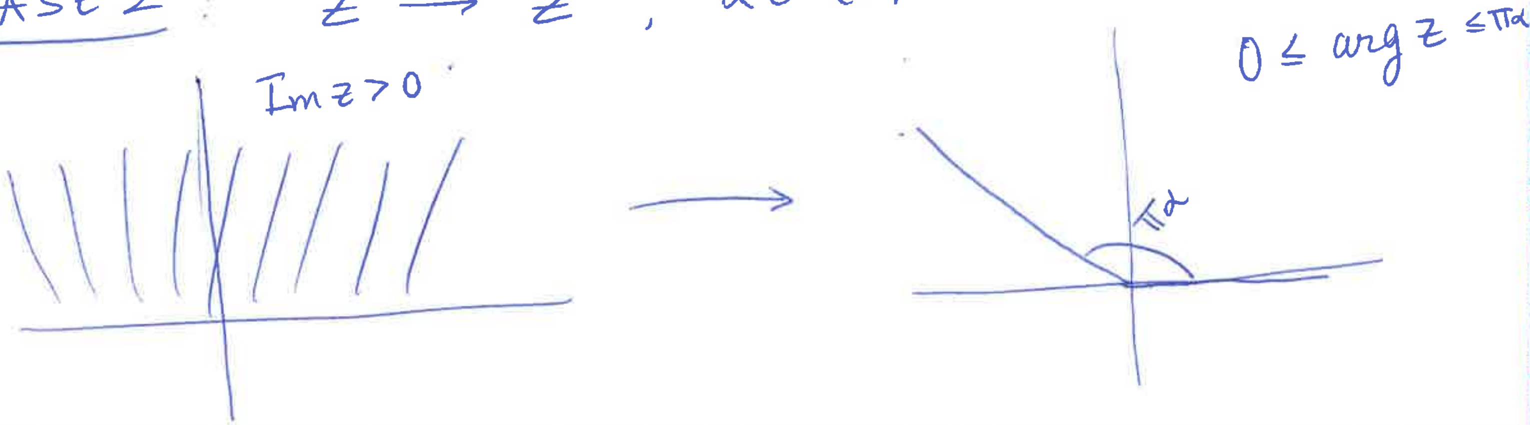
Dilate by r $\longrightarrow$ $T(z) = rz$.

2) Powers:

CASE 1 $\quad z \longrightarrow z^n$, $n \in \mathbb{N}$. Maps

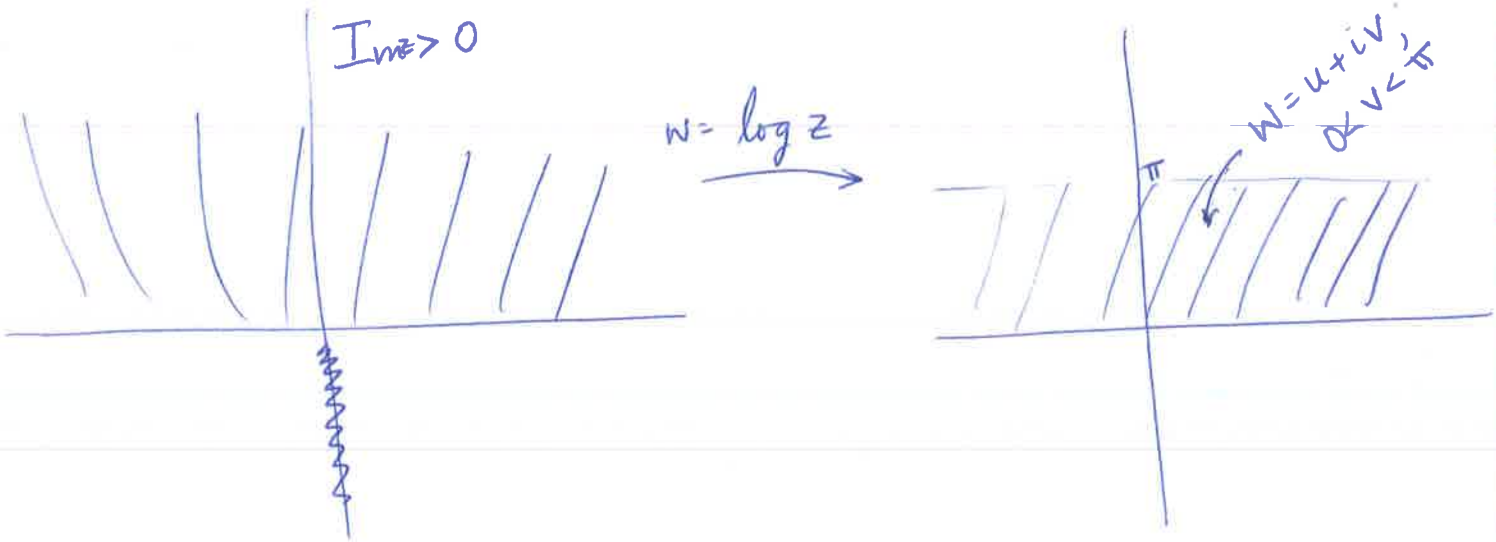


CASE 2: $\quad z \longrightarrow z^\alpha$, $\alpha \in (0,2)$.

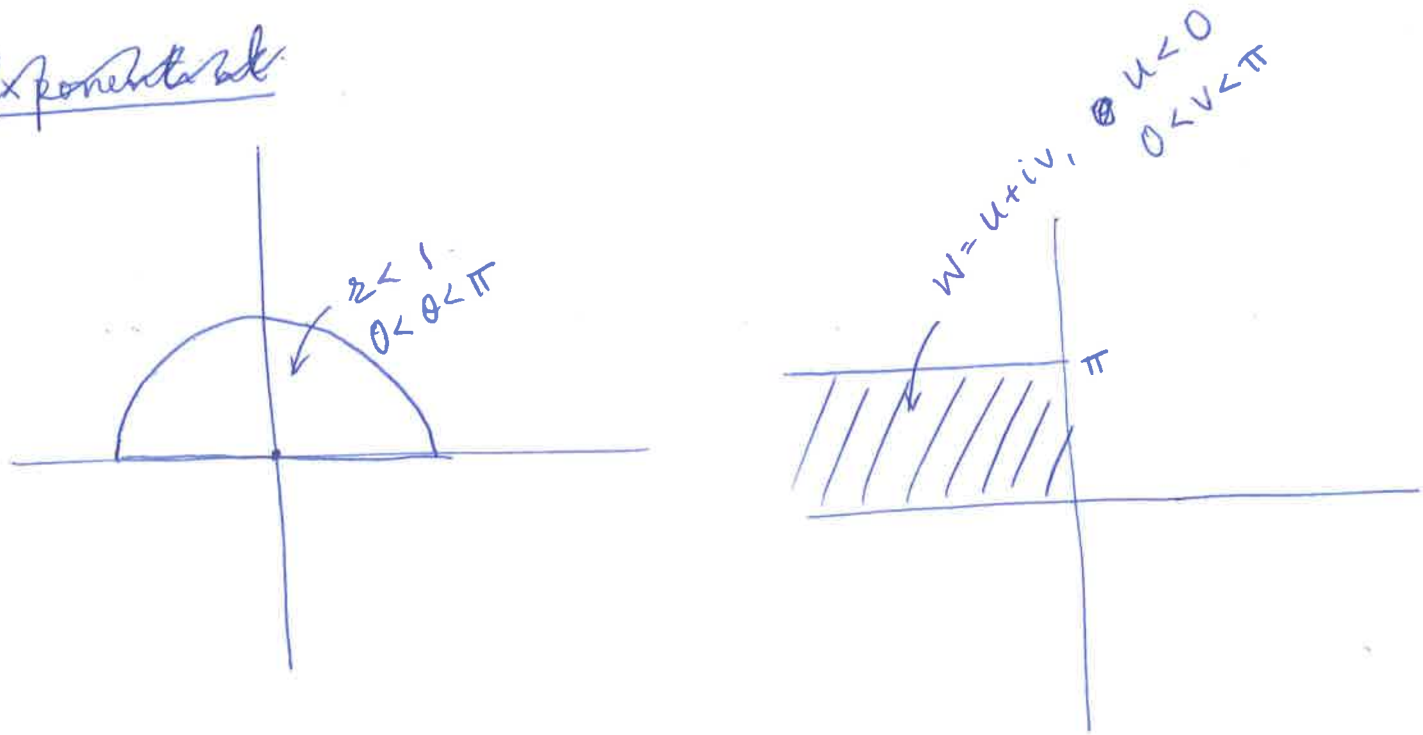


3) <u>Logarithm</u>: $z \longrightarrow \log z = \log r + i\theta$,

$z = re^{i\theta}$ with $\theta \in (-\pi/2, 3\pi/2)$. i.e we remove $\{z \mid z = ia \text{ with } a < 0\}$.

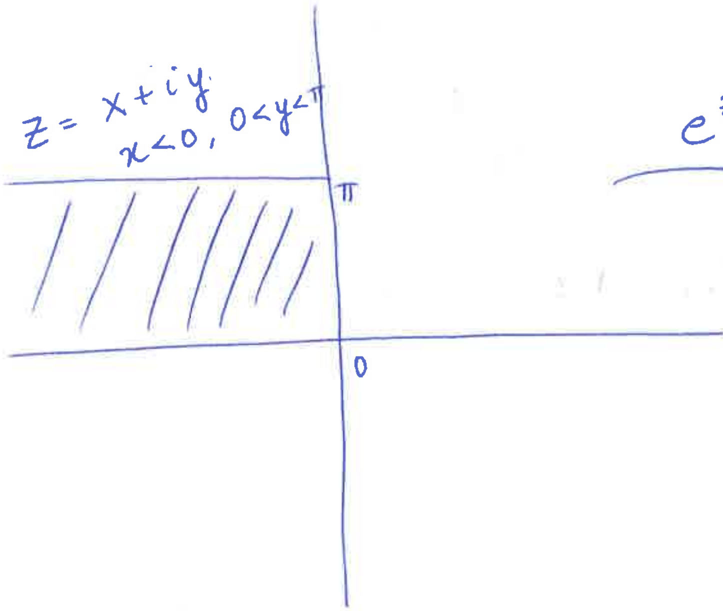


4) <u>Exponential</u>

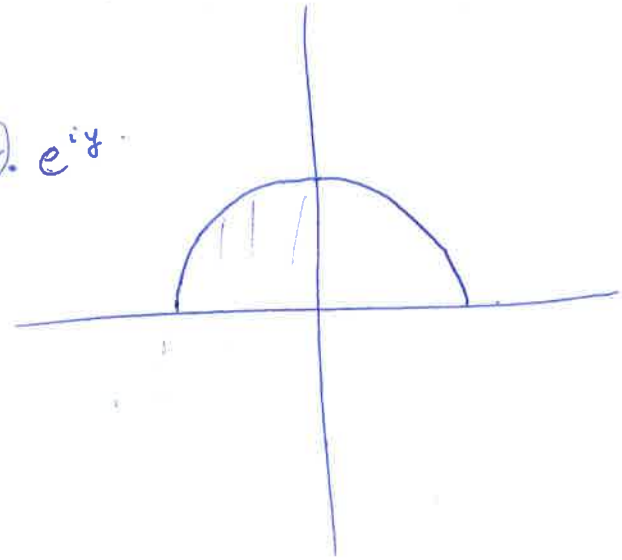


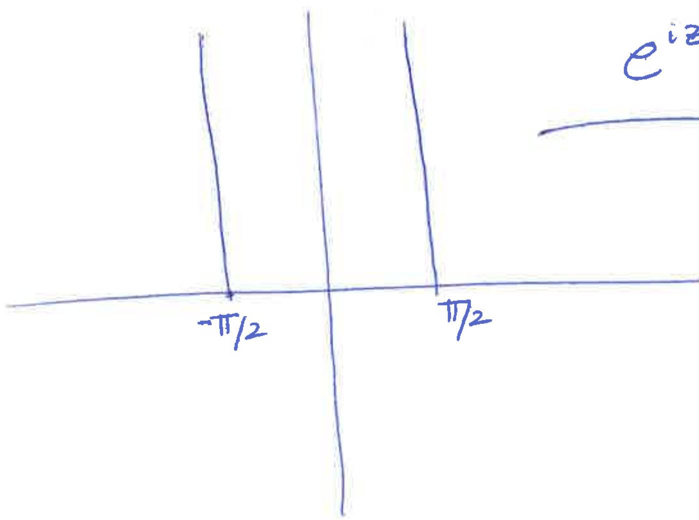
4) <u>Exponential</u>: $z \to e^{z}$.

$z = x + iy$, $x < 0$, $0 < y < \pi$

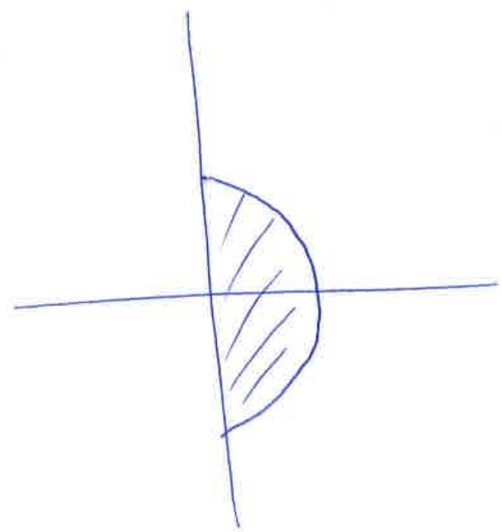


$$e^{z} = e^{x} \cdot e^{iy}$$

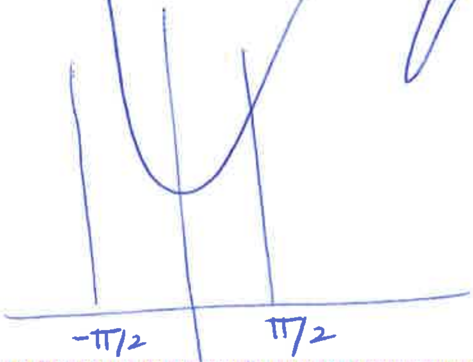




$$e^{iz} = e^{-y} \cdot e^{ix}$$



5) <u>Sine</u>: $z \to \sin z = \frac{e^{iz} - e^{-iz}}{2i}$



- <u>Riemann Mapping Theorem</u>.

<u>Thm</u>: Let $\Omega \subset \mathbb{C}$ be an open simply connected domain that is <u>NOT</u> whole of $\mathbb{C}$. Then for any $z_0 \in \Omega$, $\exists$ <u>unique</u> bi-holomorphism $F : \Omega \longrightarrow \mathbb{D} = \{z \mid |z| < 1\}$ s.t

$$F(z_0) = 0, \quad F'(z_0) \text{ is real \& positive}.$$

<u>Cor</u>: Any 2 <u>proper</u> simply connected subsets of $\mathbb{C}$ are bi-holomorphic.

<u>Rk</u>: NOT true if $\Omega = \mathbb{C}$ by Liouville.

- <u>Schwarz Lemma</u>: $f : \mathbb{D} \longrightarrow \mathbb{D}$ hol s.t $f(0) = 0$.

Then $$|f(z)| \le |z|.$$

Moreover if $|f(z)| = |z|$, for some $z \neq 0$ or $|f'(0)| = 1$, then $f(z) = az$ for some $|a| = 1$.