HOMEWORK-1

- pg 24 - Exercise 1
- pg 26 - Exercise 7

Additional problems.

(1) Find solutions to
$$z^2 = i.$$
Hint/Solution. Write $z = x + iy$, then we get two equations
$$x^2 - y^2 = 0$$
$$2xy = 1$$

(2) For any integer $n > 0$, Find solutions to
$$z^n = 1.$$
The solutions are called the n^{th} roots of unity.
Answer. There are n solutions given by $z = e^{\frac{2\pi ik}{n}}$, $k = 0, 1, \cdots, n-1$.

(3) Show that there are complex numbers z satisfying
$$|z-a| + |z+a| = 2|c|$$
if and only if $|a| \leq |c|$. If this condition is satisfied, what are the smallest and largest values of $|z|$.
Hint/Solution. By triangle inequality
$$2|a| = |a - z + z + a| \leq |z-a| + |z+a| \leq 2|c|.$$
The figure represented by the equation is an ellipse with major axis of length $2|c|$. Since the largest value of $|z|$ does not change if we rotate the figure, we can assume that a is a positive real number. Then the maximum value is when z is on the major axis and on the ellipse, and so maximum value of $|z|$ is $|c|$.

(4) Let $\omega, \tau \in \mathbb{C}\backslash\{0\}$ such that ω/τ is not real. A function is said to be *doubly periodic* with periods ω and τ if
$$f(z+\omega) = f(z), \text{ and } f(z+\tau) = f(z)$$
for all z. Show that a doubly periodic continuous function $f : \mathbb{C} \to \mathbb{C}$ is bounded.
Hint. Because of the periodicity, the values of $f(z)$ is decided by the values on the closure of the parallelogram spanned by ω and τ. But this is compact, and so continuity implies that the image has to be compact, and in particular bounded.

(5) From the definition of holomorphicity, show that $f(z)$ and $\overline{f(\bar{z})}$ are simultaneously holomorphic.
Hint/Solution. Let $f(z)$ be holomorphic at p, then we show that $g(z) = \overline{f(\bar{z})}$ is holomorphic at $\bar{p}$.

$$\begin{aligned}
\lim_{h\to 0} \frac{g(\bar{p}+h) - g(\bar{p})}{h} &= \lim_{h\to 0} \frac{\overline{f(\overline{\bar{p}+h})} - \overline{f(\bar{\bar{p}})}}{h} \\
&= \lim_{h\to 0} \overline{\frac{f(p+\bar{h}) - f(p)}{\bar{h}}} \\
&= \lim_{\bar{h}\to 0} \overline{\frac{f(p+\bar{h}) - f(p)}{\bar{h}}} \\
&= \overline{f'(p)}.
\end{aligned}$$

And so g is differentiable at $\bar{p}$ with $g'(\bar{p}) = \overline{f'(p)}$.