WEEK - 7

Ch 7 : Arzela - Ascoli Th^m

- **Ques**: Given a seq^n $f_n : E \to \mathbb{R}$, when is there a sub-seq^n $\{f_{n_k}\}$ s.t $f_{n_k} \xrightarrow{u.c} f$, for some f?

Example $f_n(x) = x^n$ on $(0,1)$. Recall $f_n \to 0$ on $(0,1)$ but NOT uniformly.

Claim: No sub-seq^n converges uniformly.

Pf: Sps $f_{n_k} \xrightarrow{u.c} f$, then $f \equiv 0$ on $(0,1)$. But

$$M_{n_k} = \sup_{(0,1)} |f_{n_k} - f| = \sup_{(0,1)} x^{n_k} = 1.$$

Since $\lim_{k \to \infty} M_{n_k} \neq 0$, contradiction!

To answer the question, we need to introduce two concepts

Def^n: A family $\mathcal{F}$ of functions on $E \subset \mathbb{R}$ is said to be

(1) pointwise bounded if $\forall\, x \in E$ $\exists$ $M(x) > 0$ s.t

$$|f(x)| < M(x) \quad \forall\, f \in \mathcal{F}.$$

(2) uniformly bounded if $\exists$ $M > 0$ s.t

$$|f(x)| < M \quad \forall\, x \in E, \quad \forall\, f \in \mathcal{F}.$$

Ex. 1) $\mathcal{F} = \{ f_n(x) = x^n,\ n = 1, 2, \cdots \}$ on $E = (0,1)$.

Then $|f_n(x)| \le 1 \quad \forall\, x \in (0,1),\ \forall\, f_n \in \mathcal{F}$. So $\mathcal{F}$ is uniformly bounded.

2) $\mathcal{F} = \{ f_n : [-1,1] \longrightarrow \mathbb{R} \mid f_n(x) = n(1-x^2)^n \}$. If $x = 0$, then $f_n(0) = n$, which is unbounded. So the family $\mathcal{F}$ is unbounded at $x = 0$.

3) $\mathcal{F} = \{ f_n : [0,1] \longrightarrow \mathbb{R} \mid f_n(x) = n^2 x^n (1-x) \}$

Since $\lim_{n\to\infty} n^2 x^n = 0 \quad \forall\, x \in [0,1)$ & $f_n(1) = 0$.

$\Longrightarrow \forall\, x \in [0,1]$, $\{f_n(x)\}$ is bounded.

Claim: $\{f_n\}$ is NOT uniformly bounded on $[0,1]$

Pf: 1st derivative test $\Longrightarrow$ max of f_n at $x_n = n/n+1$

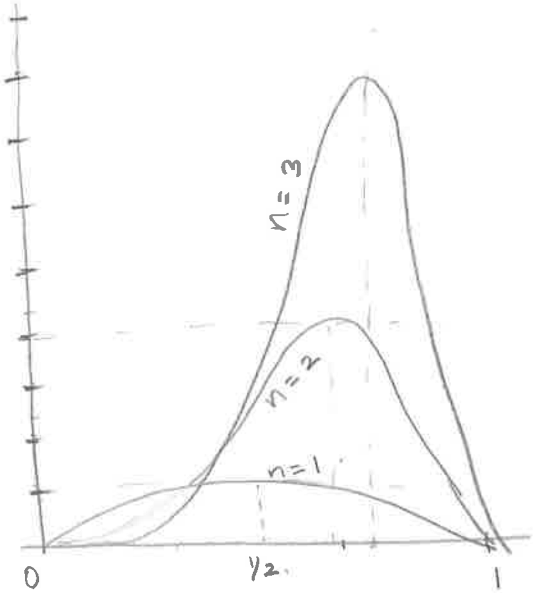
$$f_n\left(\frac{n}{n+1}\right) = n^2 \left(\frac{n}{n+1}\right)^n \left(1 - \frac{n}{n+1}\right)$$

$$= n \left(\frac{n}{n+1}\right)^{n+1} = n \left(1 - \frac{1}{n+1}\right)^{n+1}$$

$$\xrightarrow{n\to\infty} \infty, \quad \text{since } \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right)^{n+1} = e^{-1}$$

So $\mathcal{F}$ is NOT uniformly bounded.

Plots.



Rk. Clearly if $f_n \longrightarrow f$ on E, then $\{f_n\}$ is pointwise bounded since convergent sequences of real numbers are bounded.

Th^m 7.1: Sps $f_n \xrightarrow{u\cdot c} f$ on E, and $\sup_E f_n$ is finite for each n, then f_n is uniformly bounded

Pf: Let M_n s.t

$$|f_n(x)| \leq M_n \quad \forall \quad x \in E.$$

$f_n \xrightarrow{u\cdot c} f \implies \exists \; N$ s.t $\forall \; n \geq N, \; \forall x \in E$

$$|f_n(x) - f(x)| \leq 1. \qquad (*)$$

In particular, $|f(x)| \leq |f(x) - f_N(x)| + |f_N(x)|$

$$\leq 1 + M_N. \qquad (**)$$

Also, $\forall \; n > N, \; \forall \; x \in E,$

$$|f_n(x)| \le |f_n(x) - f(x)| + |f(x)|$$
$$\le 2 + M_N.$$

Let $M = \max(M_1, M_2, \dots, M_{N-1}, 2 + M_N)$.

Then $\forall\, x \in E$, $\forall\, n$,
$$|f_n(x)| \le M.$$

Rk So if $\exists$ sub-seq^n s.t $f_{n_k} \xrightarrow{u.c} f$, then a neccessary condition is that $\{f_{n_k}\}$ is uniformly bounded.

Def^n: A family of functions $\mathcal{F}$ on $E \subset \mathbb{R}$ is called equicontinuous if $\forall\, \varepsilon > 0$, $\exists\, \delta = \delta(\varepsilon) > 0$

s.t

$$\left.\begin{array}{l} |x - y| < \delta \\ x, y \in E \\ f \in \mathcal{F} \end{array}\right\} \implies |f(x) - f(y)| < \varepsilon.$$

Rk: In particular, if $\mathcal{F}$ is an equicont. family, then every $f \in \mathcal{F}$ is uniformly cont.

Ex: Let $C^1_M[a,b] = \{ f : [a,b] \longrightarrow \mathbb{R} \mid f$ cont. on $[a,b]$, diff on (a,b) with $|f'(t)| < M\}$

For any $t, s \in [a,b]$ by MVT,

$$|f(s) - f(t)| \leq M|s-t|.$$

So given $\varepsilon > 0$, let $\delta = \varepsilon/M$. Then $\forall f \in C^1_M[a,b]$ $\forall s,t \in [a,b]$

$$|s-t| < \delta \implies |f(s) - f(t)| < \varepsilon.$$

So $C^1_M[a,b]$ is an equicont. family.

More generaly, let

$$Lip_M[a,b] = \{f : [a,b] \to \mathbb{R} \mid |f(x) - f(y)| \leq M|x-y| \ \forall x,y \in [a,b]\}$$

then $Lip_M[a,b]$ is an equicont. family.

2) $f_n(x) = \dfrac{\sin(nx)}{\sqrt{n}}$

Claim: $\{f_n\}$ is an equicont. family on $\mathbb{R}$.

Pf: Note

$$|f_n(x) - f_n(y)| = \frac{1}{\sqrt{n}}|\sin(nx) - \sin(ny)|$$

$$\leq \frac{2}{\sqrt{n}}.$$

Pick N s.t $2/\sqrt{N} < \varepsilon$. Then $\forall n > N$,

$$|f_n(x) - f_n(y)| < \varepsilon \quad (\forall x,y \in \mathbb{R}). \quad (*)$$

For $n = 1, \dots, N$, note that

$$|f_n'(x)| = |\sqrt{n}\sin nx| \leq \sqrt{N}.$$

Given $\varepsilon > 0$, let $\delta = \varepsilon/\sqrt{N}$. Then $\forall\ n < N$,

$$|x - y| < \delta \implies |f_n(x) - f_n(y)| < \varepsilon.$$

$(*) \implies \forall\ n$;

$$|x - y| < \delta \implies |f_n(x) - f_n(y)| < \varepsilon.$$

So $\{f_n\}$ is an equicont. family.

Rk: So uniformly bounded derivative is not a requirement for equicont.

3) Consider $f_n(x) = x^n$ on $[0,1]$.

Claim f_n is NOT equicont.

Pf: Goal: $\exists\ \varepsilon > 0$ & N s.t $\forall n > N$, $\exists\ x_n, y_n \in (0,1)$ s.t

$$|x_n - y_n| \leq \frac{1}{n} \quad \text{but} \quad |f_n(x_n) - f_n(y_n)| \geq \varepsilon.$$

Let $x_n = 1 - \frac{1}{n}$, $y_n = 1$. Then

$$|f_n(x_n) - f_n(y_n)| = 1 - \left(1 - \frac{1}{n}\right)^n \xrightarrow{n \to \infty} 1 - e^{-1} > 0$$

Let $\varepsilon = (1 - e^{-1})/2$. Then $\exists\ N$ s.t. $\forall\ n > N$

$$|f_n(x_n) - f_n(y_n)| > \frac{1 - e^{-1}}{2}.$$

Done!

Thm 7.2 Let $f_n : [a,b] \longrightarrow \mathbb{R}$ cont. s.t $f_n \xrightarrow{u.c} f$ on $[a,b]$. Then $\{f_n\}$ is equicont.

Pf: ($\varepsilon/3$ - trick). Let $\varepsilon > 0$.

$f_n \xrightarrow{u.c} f \underset{\text{Cauchy)}}{\overset{\text{(uniform}}{\Longrightarrow}} \exists\ N$ s.t $\forall\ x \in [a,b]$,

$$|f_n(x) - f_N(x)| < \frac{\varepsilon}{3} \quad \forall\ n > N. \quad (*)$$

Claim $\exists\ \delta_N$ s.t.

$$\left.\begin{array}{l} |x - y| < \delta_N \\ n > N \\ x, y \in [a,b] \end{array}\right\} \Longrightarrow |f_n(x) - f_n(y)| < \varepsilon.$$

Pf: f_N is uniformly cont. on $[a,b]$. So, $\exists$ $\delta_N > 0$ s.t

$$\left.\begin{array}{l} |x - y| < \delta_N \\ x, y \in [a,b] \end{array}\right\} \Longrightarrow |f_N(x) - f_N(y)| < \frac{\varepsilon}{3}.$$

If $|x - y| < \delta_N$ & $n > N$, then

$$\begin{aligned} |f_n(x) - f_n(y)| &\leq |f_n(x) - f_N(x)| + |f_N(x) - f_N(y)| \\ &\quad + |f_N(y) - f_n(y)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Done!

Now, for $k = 1, 2, \dots, N$ f_k is u. cont on $[a,b]$. So, $\exists$ $\delta_k > 0$ s.t

$$|x - y| < \delta_k \implies |f_k(x) - f_k(y)| < \varepsilon$$

Let $\delta = \min(\delta_1, \delta_2, \dots, \delta_{N-1}, \delta_N) > 0$.

Then

$$|x - y| < \delta \implies |f_n(x) - f_n(y)| < \varepsilon \quad \forall n$$

<u>Rk</u>: Proof works if $[a,b]$ is replaced by an arbitrary subset $E \subset \mathbb{R}$ if each f_n is assumed to be uniformly cont.

<u>Th^m</u> 7.3 (Arzela-Ascoli) Let $f_n : [a,b] \to \mathbb{R}$ be a <u>pointwise</u> <u>bounded</u> and <u>equicont</u>. seq^n of functions. Then

(1) $\{f_n\}$ is uniformly bounded.

(2) $\exists$ sub-seq^n $\{f_{n_k}\}$ which converges uniformly.

The proof has several steps. The <u>key</u> ingredeint is the foll.

<u>Lemma</u> Let $E = \{p_1, p_2, \dots\}$ be a seq^n of points, and $f_n : E \to \mathbb{R}$ be <u>pointwise bounded</u>

Then $\exists$ sub-seq^n $\{f_{n_k}\}$ which converges point wise on E.

Pf: (Cantor diagonalization). Consider the seq^n $\{f_n(p_1)\}$. This is a bounded seq^n. So Bolzano-Weierstrass $\overset{(BW)}{\Longrightarrow}$ $\exists$ sub-seq^n, which we denote by $\{f_{1,j}\}$ s.t.

$$\lim_{j\to\infty} f_{1,j}(p_1) \text{ exists \& finite.}$$

Now, consider seq^n $\{f_{1,j}(p_2)\}$. This is again bounded. So BW $\Longrightarrow$ $\exists$ sub-seq $\{f_{2,j}\}$ s.t $\{f_{2,j}(p_2)\}$ conv. Continuing we construct sequences $S_1, S_2, \ldots$

$$S_1: f_{1,1}, f_{1,2}, f_{1,3}, \ldots$$

$$S_2: f_{2,1}, f_{2,2}, f_{2,3} \ldots$$

$$S_3: f_{3,1}, f_{3,2}, f_{3,3}, \ldots$$

$$\vdots$$

$$S_k: f_{k,1}, f_{k,2}, f_{k,3}, \ldots$$

$$\vdots \quad \ldots$$

s.t

(1) $S_k \subset S_{k-1}$ i.e $S_k = \{f_{k,j}\}_{j=1}^{\infty}$ is a sub-seq^n of $S_{k-1}, \ldots, S_1 \subset \{f_n\}$.

(2) $\{f_{k,j}(P_k)\}$ converges as $j \to \infty$.

Now, consider the "diagonal"

$$D = \{f_{1,1}, f_{2,2}, f_{3,3}, \ldots \} = \{f_{k,k}\}.$$

Clearly $\{f_{k,k}\}$ is a sub-seq^n of $\{f_n\}$.

<u>Claim</u>: $\{f_{k,k}(p_m)\}$ converges as $k \to \infty$ $\forall$ $p_m \in E$.

<u>Pf</u>: Given p_m, $\{f_{k,k}\}_{k \geq m}$ is a sub-seq^n of S_m.

Since $f_{m,j}(p_m)$ converges $\implies$ $\{f_{k,k}(p_m)\}$ also converges.

- <u>Proof of Arzela - Ascoli</u>

(1) $\{f_n\}$ equicont. So $\exists$ $\delta > 0$ s.t

$|x - y| < \delta \implies |f_n(x) - f_n(y)| < 1$ $\forall$ n.

Let $p_1 < p_2 < \cdots < p_\ell$ s.t

s.t $p_i - p_{i-1} < \delta$, $p_1 - a < \delta$, $b - p_\ell < \delta$.

Then for any $x \in [a, b]$, $\exists$ p_k s.t $|x - p_k| < \delta$.

f_n pointwise bdd $\Longrightarrow$ $\forall\ i = 1, \ldots, l$, $\exists\ M_i$ s.t

$$|f_n(p_i)| < M_i \quad \forall\ n.$$

If $x \in [a,b]$, let p_i s.t $|x - p_i| < \delta$. Then

$$|f_n(x)| \leq |f_n(x) - f_n(p_i)| + |f_n(p_i)|$$

$$\leq 1 + M_i$$

So if $M = \max(M_1, M_2, \ldots, M_l) + 1$, then

$$|f_n(x)| \leq M \quad \forall\ n,\ \forall\ x \in [a,b].$$

$\Longrightarrow$ $\{f_n\}$ uniformly bounded.

(2) Let $E = [a,b] \cap \mathbb{Q}$

<u>FACT</u> E is countable i.e $E = \{p_1, p_2, \ldots\}$ is a seq^n.

Lemma $\Longrightarrow$ $\exists$ sub-seq^n f_{n_k} s.t $f_{n_k}(p_m)$ conv. $\forall\ p_m \in E$.

<u>Claim</u> $\{f_{n_k}\}$ conv. uniformly on $[a,b]$.

<u>Pf</u>: For simplicity of notation, let $g_k = f_{n_k}$.

Let $\varepsilon > 0$.

<u>Goal</u>: $\{g_k\}$ is uniformly Cauchy.

$\{g_k\}$ equicont. $\Longrightarrow$ $\exists\ \delta > 0$ s.t. $\forall k$,

$$|x - y| < \delta \Longrightarrow |g_k(x) - g_k(y)| < \varepsilon/3 \quad (*)$$

$\exists\ p_1, \dots, p_\ell \in E$ s.t.

$$p_i - p_{i-1} < \delta, \quad p_1 - a < \delta, \quad b - p_\ell < \delta.$$

$\{g_k(p_i)\}$ conv. for $i = 1, \dots, \ell$. $\Longrightarrow$ $\exists\ N_i$ s.t.

if $j, k > N_i \Longrightarrow |g_j(p_i) - g_k(p_i)| < \varepsilon/3 \quad (**)$.

Let $N = \max(N_1, \dots, N_\ell)$.

Now, for $x \in [a, b]$, let p_i s.t. $|x - p_i| < \delta$.

Then for all $j, k > N$

$$|g_j(x) - g_k(x)| \le |g_j(x) - g_j(p_i)| + |g_j(p_i) - g_k(p_i)| + |g_k(x) - g_k(p_i)|$$

$$\underset{(**)}{\overset{(*)}{<}} \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

So Given $\varepsilon > 0$, we have found N s.t.

$$\left.\begin{array}{l} j, k > N \\ x \in [a, b] \end{array}\right\} \Longrightarrow |g_j(x) - g_k(x)| < \varepsilon.$$

So $\{g_k\}$ is uniformly Cauchy, and hence convergent.

Cor 7.4. Let $f_n : [a,b] \longrightarrow \mathbb{R}$ s.t

(1) $f_n \longrightarrow f$ on $[a,b]$

(2) $\{f_n\}$ is an equicont. seq^n.

Then $f_n \xrightarrow{u.c} f$.

Pf: Sps not. Then $\exists\ \varepsilon > 0$ and sub-sequence $\{f_{n_k}\}$ s.t.

$$\sup_{[a,b]} |f_{n_k}(x) - f(x)| \geq \varepsilon \quad \forall\ k \quad (*)$$

By (1) $f_{n_k} \xrightarrow{k\to\infty} f$, so $\{f_{n_k}\}$ is pointwise bounded

(2) $\Longrightarrow \{f_{n_k}\}$ also equicont.

Arzela-Ascoli $\Longrightarrow \exists$ sub seq of f_{n_k}, say $\{f_{n_{k_j}}\}$ which converges uniformly.

But then $f_{n_k} \longrightarrow f \Longrightarrow f_{n_{k_j}} \xrightarrow[j\to\infty]{u.c} f$.

So $\exists\ J$ s.t

$$\sup_{[a,b]} |f_{n_{k_J}}(x) - f(x)| < \varepsilon.$$

contradicting $(*)$ for $k = k_J$.

Examples: Let f_n be a seq^n defined by

$$f_n(x) = \int_0^x e^{-t^2} \sin(nt)\, dt$$

on $[0,1]$. Then since $|e^{-t^2}\sin(nt)| \le 1$ on $[0,1]$

$$\Rightarrow \quad |f_n(x)| \le \int_0^x dt = x \le 1 \quad \forall x \in [0,1]$$

So $\{f_n\}$ uniformly bounded.

Also $e^{-t^2}\sin(nt)$ cont. So fund. Th^m $\Rightarrow$

$$f_n'(x) = e^{-x^2}\sin(nx)$$

So $|f_n'(x)| \le 1$ on $(0,1)$. So MVT $\Rightarrow$

$$|f_n(x) - f_n(y)| \le |x-y|.$$

$\Rightarrow$ $\{f_n\}$ is an equicont. family.

Arzela-Ascoli $\Rightarrow$ $\exists$ conv. subsequence $\{f_{n_k}\}$.

CHALLENGE Show $f_n \xrightarrow{u.c} 0$ on $[0,1]$.

Rk: Arzela-Ascoli is a fundamental tool in proving convergence of some numerical schemes to solve diff. equations. See Assignment for an elementary example

Ch. 8 Metric Spaces

Def^n. Let X be a set. A metric or distance function on X is a function

$$d : X \times X \longrightarrow \mathbb{R}.$$
$$(x, y) \longrightarrow d(x, y).$$

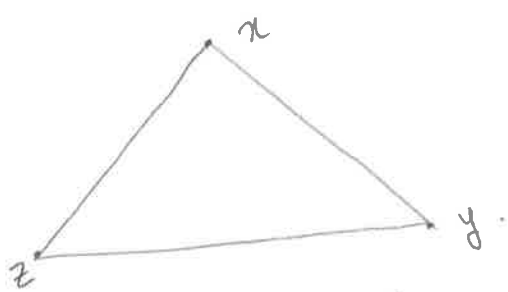
s.t.

D1 (Positive definiteness) $d(x, y) > 0$ if $x \neq y$ and $d(x, x) = 0$.

D2 (Symmetry) $d(x, y) = d(y, x) \quad \forall \; x, y \in X$.

D3 (Triangle ineq). $\forall \; x, y, z \in X$.

$$d(x, y) \leq d(x, z) + d(y, z)$$



We then call (X, d) a metric space, and refer to elements of X as points.

Examples 1) $(\mathbb{R}, |\cdot|)$ We define the distance function by

$$d(s, t) = |s - t|$$

2) <u>Euclidean spaces</u>. For $n \geq 1$, $n \in \mathbb{N}$, define

$$\mathbb{R}^n = \{ \vec{x} = (x_1, \dots, x_n) \mid x_j \in \mathbb{R} \}.$$

So, in particular $\mathbb{R}^1 = \mathbb{R}$. If $\vec{x} = (x_1, \dots, x_n)$ & $\vec{y} = (y_1, \dots, y_n)$, we define

(1) $\vec{x} + \vec{y} = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$

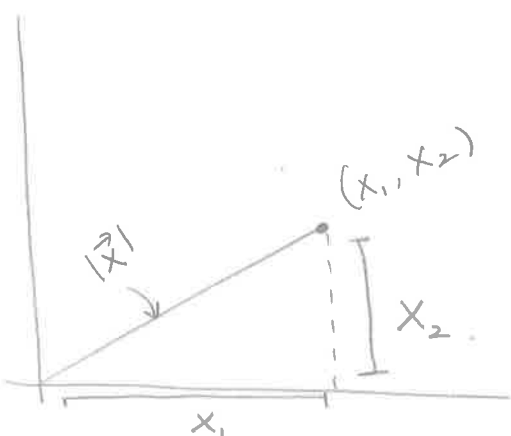
(2) $-\vec{x} = (-x_1, -x_2, \dots, -x_n)$

(3) $\vec{x} - \vec{y} = (x_1 - y_1, \dots, x_n - y_n)$

(4) (Absolute value / norm).

$$|\vec{x}| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

When $n = 2$, $|\vec{x}| = \sqrt{x_1^2 + x_2^2}$, which by Pythagoras is the distance from the origin $\vec{0} = (0, 0)$.



More generally $(\forall n)$ we define

$$\boxed{d(\vec{x}, \vec{y}) := |\vec{x} - \vec{y}|}$$

<u>Thm 8.1</u> $(\mathbb{R}^n, |\cdot|)$ is a metric space.

<u>Pf</u>: We have to show $|\cdot|$ satisfies D1 - D3. D1 & D2 are obvious. For D3, we first claim

<u>Claim</u> For $\vec{u}, \vec{v} \in \mathbb{R}^n$,

$$|\vec{u} + \vec{v}| \leq |\vec{u}| + |\vec{v}|.$$

Applying this to $\vec{u} = \vec{x} - \vec{z}$, $\vec{v} = \vec{z} - \vec{y}$

$$\begin{aligned} d(\vec{x}, \vec{y}) = |\vec{x} - \vec{y}| &= |\vec{u} + \vec{v}| \\ &\overset{(\text{Claim})}{\leq} |\vec{u}| + |\vec{v}| \\ &= |\vec{x} - \vec{z}| + |\vec{z} - \vec{y}| \\ &= d(x, z) + d(y, z). \end{aligned}$$

<u>Pf of Claim</u>: Let $\vec{u} = (u_1, \ldots, u_n)$, $\vec{v} = (v_1, \ldots, v_n)$.

The proof relies on the <u>Cauchy - Schwarz</u> (C-S) inequality

$$\left| \sum_{i=1}^{n} u_i v_i \right| \leq \left(\sum_{i=1}^{n} u_i^2 \right)^{1/2} \left(\sum_{j=1}^{n} v_j^2 \right)^{1/2} = |\vec{u}| |\vec{v}|.$$

(NOTE: LHS is $|\vec{u} \cdot \vec{v}|$, $n = 2$, $\vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos\theta$, so ineq follows from fact that $|\cos\theta| \leq 1$. When $n > 2$, this arg. does not work since θ is <u>defined</u> then as $\cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|} \right)$ which relies on the fact that $(\ )$ is ≤ 1 i.e the C-S ineq)

Now,

$$(|\vec{u}| + |\vec{v}|)^2 - |\vec{u} + \vec{v}|^2 = \cancel{|\vec{u}|^2} + \cancel{|\vec{v}|^2} + 2|\vec{u}||\vec{v}| - \sum_{i=1}^{n}\left(\cancel{u_i^2} + \cancel{v_i^2} + 2u_iv_i\right)$$

$$= 2\left[|\vec{u}||\vec{v}| - \sum_{i=1}^{n} u_iv_i\right] \geq 0$$

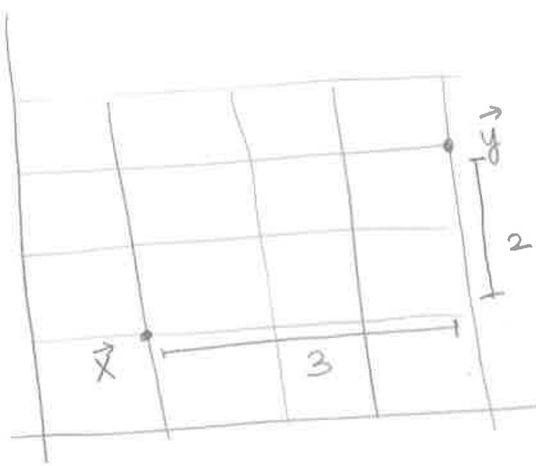
by C-S. Done!

3) We define the <u>taxicab</u> function of $\mathbb{R}^n$ by

$$d_t(\vec{x}, \vec{y}) = \sum_{i=1}^{n} |x_i - y_i|.$$

<u>Check</u>: $d_t(\vec{x}, \vec{y})$ is a metric on $\mathbb{R}^n$.

It is called <u>taxicab</u> since it gives the <u>block</u> distance from point A to point B in a city designed like a grid.



So $d(\vec{x}, \vec{y})$
$= 3 + 2 = 5.$

4) (Discrete metric). Let X be *any* set. Define

$$d_{dis}(x, y) = \begin{cases} 1, & x \neq y \\ 0, & x = y. \end{cases}$$

Check d satisfies D1 - D3.

Called discrete since each point is "isolated".

Rk: In many senses, this is the worst possible metric, and hence most useful in producing pathalogical behaviour & counter examples.

Def^n (Metric Subspace) Let (X, d) be a metric space & $Y \subset X$. Then d induces a metric d_Y on Y by restriction i.e we define

$$d_Y(x, y) = d(x, y) \quad \forall \; x, y \in Y.$$

We then call (Y, d_Y) a metric subspace of (X, d) and write $(Y, d_Y) \subset (X, d)$.

Ex: $\mathbb{Q} \subset \mathbb{R}$. Metric $|\cdot|$ on $\mathbb{R}$ induces a metric $|\cdot|_{\mathbb{Q}}$ on $\mathbb{Q}$ &

$$(\mathbb{Q}, |\cdot|_{\mathbb{Q}}) \subset (\mathbb{R}, |\cdot|).$$

• Open sets & interior points Let (X, d) be a metric space.

Defn. For $p \in X$ and $r > 0$, the (open) ball of radius 'r' around p is defined to be

$$B_r^X(p) = B_r(p) := \{x \in X \mid d(p, x) < r\}.$$

The closed ball of radius 'r' is defined to be

$$B_{\leq r}(p) := \{x \in X \mid d(p, x) \leq r\}.$$

Ex: 1) Let $X = \mathbb{R}$ with $|\cdot|$. Then

$$B_r(p) = (p - r, p + r)$$
$$B_{\leq r}(p) = [p - r, p + r].$$

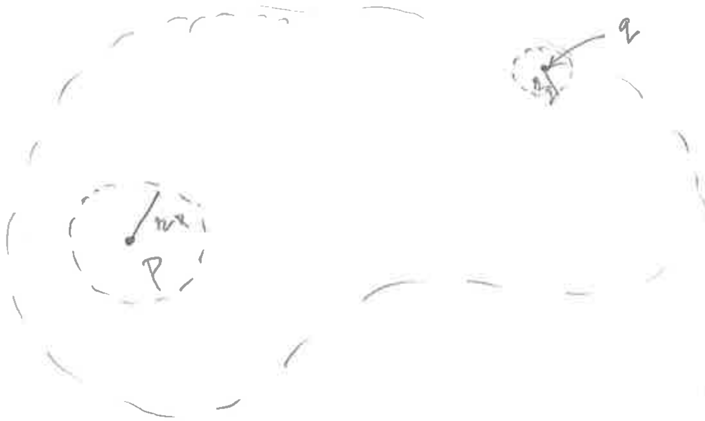
2) (X, d_{dis}): Then for any $p \in X$.

$$B_r(p) = \begin{cases} \{p\}, & r \leq 1. \\ X, & r > 1. \end{cases}$$

$$B_{\leq r}(p) = \begin{cases} \{p\}, & r < 1. \\ X, & r \geq 1. \end{cases}$$

Def^n A subset $U \subset X$ is called *open*, if $\forall\ p \in U$, $\exists\ r = r(p) > 0$ s.t.

$$B_r(p) \subset U.$$



Rk. Note that X is an open set. By *convention* the empty set $\emptyset$ is also considered open.

Ex. 1) Any open interval $(a, b) \subseteq \mathbb{R}$ is an open set in $(\mathbb{R}, |\cdot|)$. Here $a = -\infty$ or $b = \infty$ is also permitted.

2) $(0, \sqrt{2}]$ is *not* open in $(\mathbb{R}, |\cdot|)$ since $\sqrt{2} \in (0, \sqrt{2}]$, but $\forall\ r > 0$, $B_r(\sqrt{2}) = (\sqrt{2} - r, \sqrt{2} + r)$ contains points from $(0, \sqrt{2}]^c$.

However if $Y = (-1, \sqrt{2}]$ with the *subspace* metric d_Y, then $(0, \sqrt{2}]$ is *open* in (Y, d_Y).

Since in this case, for instance

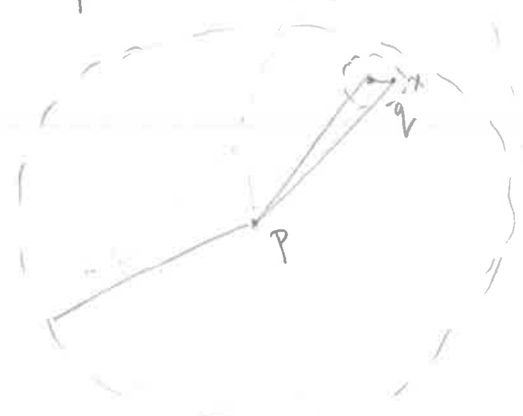
$$B_1^Y(\sqrt{2}) := \{x \in Y \mid |x - \sqrt{2}| < 1\}$$
$$= (\sqrt{2} - 1, \sqrt{2}].$$

CAUTION: Let (X, d) be a metric space & $Y \subset X$ with subspace metric d_Y. Then there might be $A \subset Y \subset X$ s.t. $A \subset_{open} (Y, d_Y)$ but A not open in (X, d).

But $A \subset_{open} X \implies A \subset_{open} Y$.

3) In (X, d_{dis}) a single point set $\{p\}$ is open since $B_{1/2}(p) \subset \{p\}$.

Th^m: $B_r(p) \subset_{open} (X, d)$ $\forall p \in X$, $\forall r > 0$.



Let $q \in B_r(p)$.

Goal: $\exists s > 0$ s.t. $B_s(q) \subset B_r(p)$.

Since $q \in B_r(p) \implies d(p, q) < r$.

Let $0 < s < r - d(p, q)$, say $s = \frac{r - d(p,q)}{2}$.

Claim: $B_s(q) \subset B_r(p)$

Pf: Let $x \in B_s(q)$. Then $d(q, x) < s$.

Δ-ineq $\Longrightarrow$

$$\begin{aligned} d(p,x) &\le d(p,q) + d(q,x) \\ &< d(p,q) + s \\ &< r. \end{aligned}$$

So $x \in B_r(p)$ and hence $B_s(q) \subset B_r(p)$.

Def^n: Let $E \subset X$. A point $p \in E$ is called an interior point of E if $\exists$ $r = r(p) > 0$ s.t $B_r(p) \subset E$. The set of interior points is called the interior, and denoted by

$$E^\circ = \{p \in E \mid \exists \; r = r(p) > 0 \text{ s.t } B_r(p) \subset E\}.$$

Th^m 8.3 1) E is open $\Longleftrightarrow$ $E^\circ = E$

2) E° is the largest open set in E i.e if $U \subset E$ & $U \subset_{open} X$, then $U \subset E^\circ$.

Pf 1) is trivial.

2) Sps $U \subset E$ & $U \subset_{open} X$, then if $p \in U$, $\exists$ $r > 0$ s.t $B_r(p) \subset U \subset E$. So $p \in E^\circ$.

So $U \subset E^\circ$.

Ex: Let $E = [-1, 1] \setminus \{0\}$. Then $E^\circ = (-1, 0) \cup (0, 1)$.

<u>Th^m 8.4</u> 1). If $\{U_\alpha\}_{\alpha\in I}$ is a collection of open sets, then $\bigcup_{\alpha\in I} U_\alpha$ is open.

2) If $U_1, \dots, U_N$ is a finite collection of open sets, then $\bigcap_{j=1}^{N} U_j$ is open.

<u>Pf</u>: 1). Let $p \in \bigcup_{\alpha\in I} U_\alpha$. Then $\exists\ \alpha_0 \in I$ s.t. $p \in U_{\alpha_0}$.

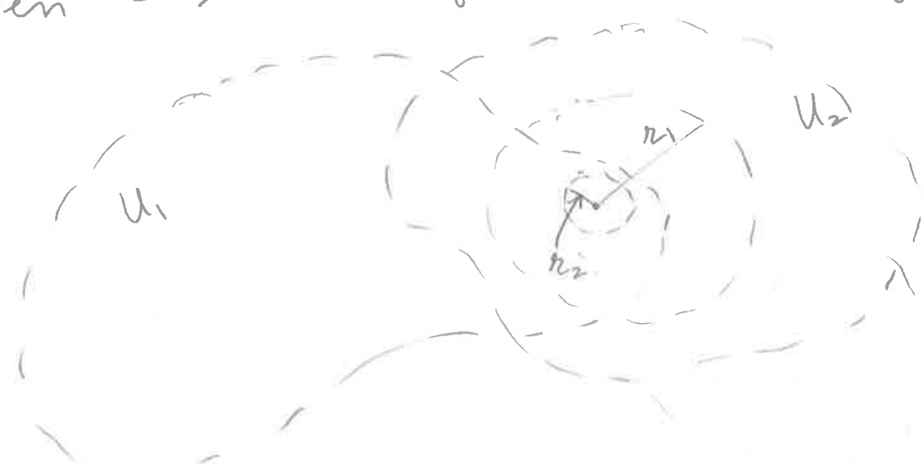
U_{α_0} open $\Longrightarrow \exists\ r > 0$ s.t

$$B_r(p) \subset U_{\alpha_0} \subset \bigcup_{\alpha\in I} U_\alpha$$

So $p \in \left(\bigcup_{\alpha\in I} U_\alpha\right)^{\circ}$. Hence $\bigcup_{\alpha\in I} U_\alpha$ open.

2) Let $p \in \bigcap_{j=1}^{N} U_j$. Then $p \in U_j \quad \forall\ j = 1, \dots, N$.

U_j open $\Longrightarrow \exists\ r_j > 0$ s.t $B_{r_j}(p) \subset U_j$



Let $r = \min(r_1, r_2, \dots, r_N) > 0$. Then.

$$B_r(p) \subset U_j \quad \forall\ j.$$

So $B_r(p) \subset \bigcap_{j=1}^{N} U_j$

Rk: The above Th^m is not true for arbitrary intersections.

e.g: Let $U_j = (-1/j, 1/j)$. Then $U_j \subset_{open} (\mathbb{R}, |\cdot|)$

But
$$\bigcap_{j=1}^{\infty} U_j = \{0\} \not\subset_{open} (\mathbb{R}, |\cdot|).$$

Ex: Consider any subset $E \subset (X, d_{dis})$. Then E is open, since $E = \bigcup_{p\in E} \{p\}$ and each $\{p\} \subset_{open} (X, d_{dis})$.

· Limit points & Closed Sets:

Def^n: Let $E \subset X$. A point $x \in X$ is called a limit point (l.p) of E if $\forall\ r>0$, $B_r(p)$ intersects E at some point other than 'x'. We denote the set of limit points by

$$L(E) := \{x \in X \mid x \text{ is a l.p of } E\}.$$

Rk: There might be points in $L(E)$ from outside E. Similarly there might be point in E, not in $L(E)$

e.g (1) $E = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 1\} \cup \{(x,0) \mid x > 0\}$.

Then $L(E) = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 1\} \cup \{(x,0) \mid x \geq 1\}$

2) $E = (0,2) \cup \{3\}$. Then $L(E) = [0,2]$

Def^n: Let $E \subset X$. A point $p \in E$ is called an isolated point of E if $\exists$ $\varepsilon > 0$ s.t $B_\varepsilon(p) \cap E = \{p\}$

The set of isolated points is

$$E \setminus L(E).$$

Ex: 1) $E = (0,2) \cup \{3\}$. Then 3 is an isolated point

2) In (X, d_{dis}) every point $p \in X$ is isolated.

Def^n: A seq^n $\{x_n\}$ in X is a function $f: \mathbb{N} \longrightarrow X$, where $x_n = f(n)$ is called the n^{th} term. We say $\{x_n\}$ is convergent if $\exists$ $p \in X$ s.t $\forall$ $\varepsilon > 0$, $\exists$ $N = N(\varepsilon) \in \mathbb{N}$ s.t

$$n > N \implies d(x_n, p) < \varepsilon.$$

We then call p the limit of $\{x_n\}$ & write

$$\lim_{n\to\infty} x_n = p \quad \text{or} \quad x_n \longrightarrow p.$$

Rk 1) $x_n \longrightarrow p \iff \forall\ \varepsilon > 0,\ \exists\ N$ s.t $x_n \in B_\varepsilon(p)$ $\forall$ $n > N$.

2) (Uniqueness) $x_n \longrightarrow p,\ x_n \longrightarrow q \implies d(p,q) = 0$
$\implies p = q$.

<u>Th^m 8.5</u>: Let $E \subset X$. Then $p \in L(E) \iff p = \lim_{n\to\infty} x_n$ for some $\{x_n\}$ in E s.t $x_n \neq p$ $\forall$ $n \in \mathbb{N}$.

<u>Pf</u>: $\Longrightarrow$ Let $p \in L(E)$. Given any $n \in \mathbb{N}$, $B_{1/n}(p) \cap E$ contains a point other than p. Let $x_n \in B_{1/n}(p) \cap E$ s.t $x_n \neq p$. Clearly $x_n \longrightarrow p$. Done!

$\Longleftarrow$ $\forall$ $\varepsilon > 0$, $\exists$ N s.t $x_N \in B_\varepsilon(p)$. Also $x_N \in E$ but $x_N \neq p$. So $\forall$ $\varepsilon > 0$ $B_\varepsilon(p) \cap E$ contains a point, namely x_N, other than p. So $p \in L(E)$.

<u>Def^n</u>. A set $A \subset X$ is called <u>closed</u> if $L(A) \subseteq A$. In particular $\emptyset$ and X are closed in X.

<u>Ex</u>. 1) $[a, b]$ is closed in $(\mathbb{R}, |\cdot|)$ for all $-\infty \leq a < b \leq \infty$.

2) Any $\{p\}$ closed (X, d) since $L(\{p\}) = \emptyset$.

<u>Th^m 8.6</u>. A closed $(X, d) \iff X \setminus A$ open (X, d).

<u>Pf</u> $\Longrightarrow$ Let $p \in A^c = X \setminus A$. A 'closed' $\Longrightarrow p \notin L(A)$. So $\exists$ $r > 0$ s.t $B_r(p) \cap A = \emptyset$. i.e $B_r(p) \subset A^c$. i.e $p \in (A^c)^\circ$. Hence A^c open.

$\Longleftarrow$ Let $p \in A^c$. A^c open $\Longrightarrow$ $\exists$ $r > 0$ s.t $B_r(p) \cap A = \emptyset$. So $p \notin L(A)$. i.e $A^c \cap L(A) = \emptyset$. Hence $L(A) \subseteq A$, and A is closed.

Cor 8.7 (1) If $\{A_\alpha\}_{\alpha\in I}$ is a collection of closed sets, then $\bigcap_{\alpha\in I} A_\alpha$ is closed.

(2) If $A_1, \dots, A_N$ is a finite collection of closed subsets, then $\bigcup_{j=1}^{N} A_j$ is closed.

Pf (1) A_α closed $\Longrightarrow$ A_α^c open

$$\Longrightarrow \bigcup_{\alpha\in I} A_\alpha^c \text{ open}$$

$$\Longrightarrow \left(\bigcup_{\alpha\in I} A_\alpha^c\right)^c = \bigcap_{\alpha\in I} A_\alpha \text{ closed}$$

(2) Follows from

$$\left(\bigcup_{j=1}^{N} A_j\right)^c = \bigcap_{j=1}^{N} A_j^c$$

Def^n The closure of $E \subset X$ is defined as

$$\overline{E} = E \cup L(E).$$

Rk: $E \subseteq \overline{E}$. So $\overline{E} \subseteq E \Longleftrightarrow L(E) \subset E$.

Th^m 8.8 (1) E closed $\Longleftrightarrow$ $E = \overline{E}$

(2) $\overline{E}$ is the smallest closed set containing E i.e. if $E \subset A$ and $A \subset_{\text{closed}} X$, then $\overline{E} \subset A$.

Pf 1) Trivial from Rk.

2) Let $E \subset A$ & $A \subset_{\text{closed}} X$, and $p \in \overline{E} \setminus E$

Then $\forall\, r > 0$, $B_r(p) \cap E \neq \phi$. Sps $p \notin A$, then
A closed $\Longrightarrow$ A^c open $\Longrightarrow$ $\exists\ r_0 > 0$ s.t
$B_{r_0}(p) \subset A^c$ i.e $B_{r_0}(p) \cap A = \phi$. But then
$E \subset A \Rightarrow B_{r_0}(p) \cap E = \phi$. Contradiction!
So $p \in A$. Hence $\bar{E} \subset A$.

Ex: If $E = [0,1)$, then $\bar{E} = [0,1]$.

Def^n. A set $E \subset X$ is called dense if $\bar{E} = X$.

Th^m 8.9 $\mathbb{Q}$ is dense in $(\mathbb{R}, |\cdot|)$.

Pf: Let $p \in \mathbb{R}$. Then $\forall\, \varepsilon$, $\exists$ a rational $q \in \mathbb{Q}$
s.t
$$p - \varepsilon < q < p$$
So $B_\varepsilon(p) \cap \mathbb{Q}$ contains some number diff from p. So $p \in L(\mathbb{Q})$. So $L(\mathbb{Q}) = \mathbb{R}$.
In particular $\bar{\mathbb{Q}} = \mathbb{R}$.

• Continuity & limits. Let (X, d_X) & (Y, d_Y) metric sp.

Def^n: Let $E \subset X$ and $f : E \longrightarrow Y$.
(1) If $p \in L(E)$, we say $\lim_{x \to p} f(x) = q$ if $\forall\, \varepsilon > 0$
$\exists\ \delta > 0$ s.t.
$$\left.\begin{array}{l} d_X(p, x) < \delta \\ x \in E \setminus \{p\} \end{array}\right\} \Longrightarrow d_Y(f(x), q) < \varepsilon$$

(2) f is said to be <u>cont.</u> at $p \in E$ if

$$\lim_{x\to p} f(x) = f(p) \iff \forall \varepsilon>0 \ \exists \delta>0 \text{ s.t } d_X(x,p)<\delta \Rightarrow d_Y(f(x),f(p))<\varepsilon$$

f is cont on E if cont. at p $\forall p \in E$.

<u>Rk</u>: If $p \in E$ is isolated, then f is cont. at p.

<u>Th^m 8.10</u> $\lim_{x\to p} f(x) = q \iff \forall \{x_n\}$ in E with $x_n \neq p \ \forall n$, $\lim_{n\to\infty} x_n = p$ we have $\lim_{n\to\infty} f(x_n) = q$.

<u>Pf</u> Exactly as proof of Th^m 3.10 (Week-3 notes). with $|\cdot|$ replaced as appropriate by d_X & d_Y.

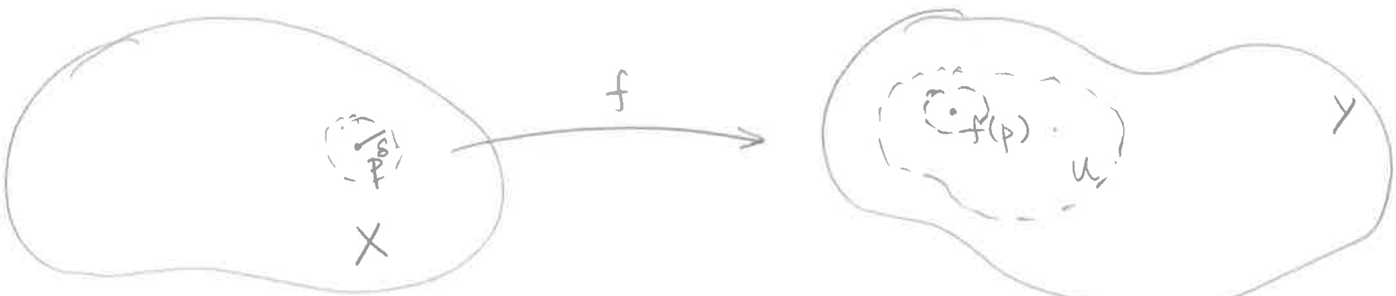
<u>Th^m 8.11</u> Let $f: X \longrightarrow Y$. TFAE

(1) f is cont. on X.

(2) $f^{-1}(U) \subset_{open} X$, whenever $U \subset_{open} Y$.

(3) $f^{-1}(A) \subset_{closed} X$ whenever $A \subset_{closed} Y$.

<u>Pf</u>: <u>(1) $\Rightarrow$ (2)</u>: Let $U \subset_{open} Y$ and $p \in f^{-1}(U)$.



Since $f(p) \in U$ & $U \subset_{open} Y$, $\exists \ \varepsilon > 0$ s.t $B_\varepsilon(f(p)) \subset U$. f cont $\Rightarrow \exists \ \delta>0$ s.t $d_X(x,p) < \delta \Rightarrow f(x) \in B_\varepsilon(f(p))$.

So, $\exists\ \delta > 0$ s.t

$$f(B_\delta(p)) \subseteq B_\varepsilon(f(p))$$

or $$B_\delta(p) \subseteq f^{-1}(B_\varepsilon(f(p))) \subseteq f^{-1}(U)$$

So p is an interior point of $f^{-1}(U)$. Since p was arbitrary $\Longrightarrow$ $f^{-1}(U)$ is open.

(2) $\Longrightarrow$ (3). Let $A \subset$ closed Y. Then $A^c \subset$ open Y.

(2) $\Longrightarrow$ $f^{-1}(A^c) \subset$ open Y. But $f^{-1}(A^c) = [f^{-1}(A)]^c$.

So $[f^{-1}(A)]^c \subset$ open $Y \xRightarrow{\text{Th}^m 8.6} f^{-1}(A) \subset$ closed X.

(3) $\Longrightarrow$ (1). Sps f is not cont at p. Then $\exists\ \varepsilon > 0$ s.t $\forall n$, $\exists\ x_n \neq p$, s.t

$$d_X(x_n, p) < \frac{1}{n}, \text{ but } d_Y(f(x_n), f(p)) \geq \varepsilon$$

Let $A = \{f(x_n) \mid n \in \mathbb{N}\}$. Then $\bar{A} \subset$ closed Y.

(3) $\Longrightarrow$ $f^{-1}(\bar{A}) \subset$ closed X. Clearly $x_n \in f^{-1}(\bar{A})\ \forall n$.

So p is a l.p of $f^{-1}(\bar{A}) \Longrightarrow p \in f^{-1}(\bar{A})$ since $f^{-1}(\bar{A})$ is closed. So $f(p) \in \bar{A}$. But since $d_Y(f(x_n), f(p)) \geq \varepsilon\ \forall\ x_n \in A$, this is a contradiction.

• Compact sets

Def^n A subset $K \subset (X, d)$ is called compact if every sequence $\{X_n\}$ in K has a sub-sequence $\{X_{n_k}\}$ that converges to a limit that is also in K

Ex 1). $(0,1]$ is not compact. Consider the sequence $X_n = 1/n$. If X_{n_k} is any sub-seq^n then either X_{n_k} is eventually const. or $X_{n_k} \longrightarrow 0$. But $0 \notin (0,1]$.

2) Claim $[a, b]$ is compact for all $-\infty < a < b < \infty$

Pf: Let $\{X_n\}$ seq^n in $[a,b]$. Bolzano-Weierstrass $\Longrightarrow$ $\exists$ conv. subseq^n $X_{n_k} \longrightarrow p$. Since $a \leq X_{n_k} \leq b$, clearly $p \in [a,b]$. So $[a,b]$ is compact by this def^n

3) (X, d) any metric space, and $K \subset X$ is a finite set. Then K is compact since Any seq^n $X_n \in K$ has a sub-seq^n that is const.

4). Consider (X, d_{dis}).

Claim $K \subset X$ is compact $\iff$ K is finite.

Pf: $\Leftarrow$ True in general.

$\Rightarrow$ Sps K is infinite. Let $\{x_n\}$ seq^n in K s.t. $x_n \neq x_m$ whenever $n \neq m$. Then $d(x_n, x_m) = 1$ $\forall\, n \neq m$. So no sub-sequence of $\{x_n\}$ can converge. So K cannot be compact. Contradiction!

Rk: It seems that compact sets are closed & do not go-off to infinity.

Def^n A subset $E \subset (X, d)$ is called bounded if $\exists$ $M > 0$ and $p \in X$ s.t.

$$d(x, p) < M \quad \forall\, x \in E.$$

$$\iff E \subset B_M(p).$$

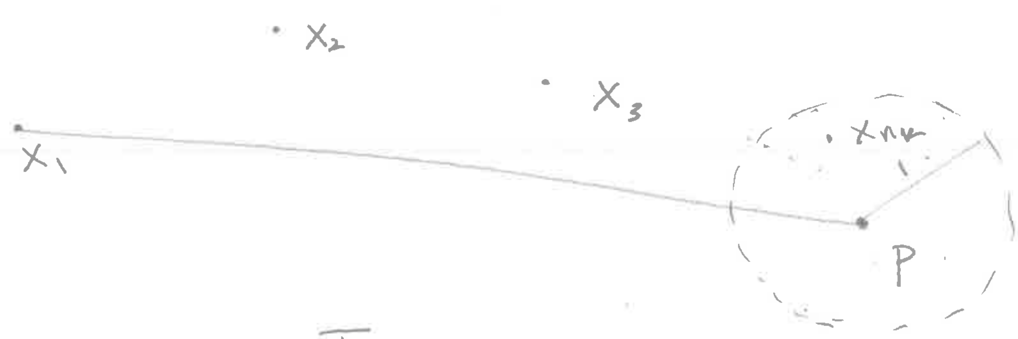
Th^m 8.12 If $K \subset (X, d)$ is compact, then K is closed & bounded.

Pf: 1) K closed: Let $p \in L(K)$. Then $\exists$ $x_n \in K$ s.t. $x_n \to p$. On the other hand. K compact $\Rightarrow$ $\exists$ sub-seq^n. $\{x_{n_k}\}$ s.t. $x_{n_k} \to q \in K$.

$x_n \to p \implies x_{n_k} \to p$. So $p = q$ and hence $p \in K$. So $L(K) \subset K$ & hence K is closed.

2) <u>K bounded</u>: If not, then $\exists$ a seq^n $x_n \in K$ s.t. $d(x_n, x_1) \geq n \quad \forall \; n > 1$.

K compact $\implies \exists$ a sub-seq^n $\{x_{n_k}\}$ s.t. $x_{n_k} \longrightarrow p \in K$.



$\exists$ J s.t. $\forall \; k > J$

$$d(x_{n_k}, p) < 1.$$

So,

$$n_k \leq d(x_{n_k}, x_1) \leq d(x_{n_k}, p) + d(x_1, p).$$

$$\leq 1 + d(x_1, p).$$

Let $k \to \infty$, then $n_k \to \infty$, but $1 + d(x_1, p)$ is a fixed finite number.

Contradiction!

Rk: The converse is not true in general.

e.g: Consider $\mathbb{Q}$ with the usual metric $|\cdot|$. Then

$$A = \{q \in \mathbb{Q} \mid 0 \leq q^2 < 2\}$$

is closed since $q \in \mathbb{Q}$ which is a l.p of A also lies in A. (note $\sqrt{2}$ is irrational).

Also A is bounded. But A is not compact since the sequence 1, 1·4, 1·41, 1·414, ... (the decimal approximations to $\sqrt{2}$) is a seq^n in A but no sub-seq. converges to any rational.

· Compact subsets of $\mathbb{R}^n$

Th^m 8·13 Let $K \subset \mathbb{R}^n$. Then K is compact $\Longleftrightarrow$ K is closed & bounded.

Key technical tool is the Bolzano-Weierstrass for bounded sets in $\mathbb{R}^n$

Th^m 8·14 (Bolzano-Weierstrass) Any bounded seq^n in $\mathbb{R}^n$ has a convergent sub-seq^n.

This follows from BW for $l = 1$ (which we proved in the first week) & the foll. obs.

Lemma 8.15 Let $\{\vec{X}_k\}$ be a sequence in $\mathbb{R}^n$ s.t $\vec{X}_k = (X_{k,1}, X_{k,2}, \ldots, X_{k,n})$. Then $\vec{X}_k \longrightarrow \vec{a} = (a_1, a_2, \ldots, a_n) \iff \lim_{k\to\infty} X_{k,j} = a_j$.

Pf of Th^m 8.13 $\implies$ Holds for any metric sp. by Th^m 8.12

$\impliedby$ Let $\{\vec{X}_k\}$ any seq^n in K. Since K is bounded $\implies$ $\{\vec{X}_k\}$ is bounded

$\overset{BW}{\implies}$ $\exists$ a convergent subsequence $\{\vec{X}_{k_j}\}$.

Suppose $\vec{X}_{k_j} \longrightarrow \vec{a} \in \mathbb{R}^n$, then $\vec{a} \in L(K)$.

K closed $\implies$ $\vec{a} \in K$.

So any seq $\vec{X}_k$ in K has a sub-seq converging to an element in K.

So K compact.

· Compact Sets and continuity

Th^m 8.16 $f: K \to Y$ cont. and $K \subset X$ compact.

Then $f(K)$ is compact in Y.

Pf: Let $y_n = f(x_n)$ be any seq^n in $f(K)$.

K compact $\implies$ $\exists$ sub-sequence $\{x_{n_k}\}$ s.t

$x_{n_k} \longrightarrow p \in K$.

Then f cont $\Longrightarrow \lim_{k\to\infty} f(x_{n_k}) = f(p) \in f(K)$.

So $y_{n_k} = f(x_{n_k}) \longrightarrow f(p) \in f(K)$.

So given any $\{y_n\}$ in $f(K)$, there is a sub-seq^n $y_{n_k} \longrightarrow q \in f(K)$.

So $f(K)$ is compact.

<u>Cor 8.17</u> (Extremum value theorem) Let $f: K \to \mathbb{R}$ be cont. and suppose $K \subset X$ is compact. Then

(1) f is bounded.

(2) $\exists$ $p, q \in K$ s.t

$$f(p) = \inf_{x\in K} f(x), \quad f(q) = \sup_{x\in K} f(x).$$

<u>Lemma 8.18</u> If $E \subset \mathbb{R}$ is compact, then $\sup E$ and $\inf E$ are finite and contained in E.

<u>Pf</u>: E compact $\Longrightarrow$ E bounded. So $\sup E < \infty$

Let $x_n \in E$ seq s.t $x_n > \sup E - \frac{1}{n}$.

E compact $\Rightarrow$ $\exists$ sub-seq s.t $x_{n_k} \longrightarrow p \in E$.

So $p \geq \sup E$. Since $p \in E$, $p \leq \sup E$.

So $p = \sup E$ & hence $\sup E \in E$.

Similar for $\inf E$.

Pf of Th^m 8.17. Th^m 8.16 $\Rightarrow$ $f(K) \subset \mathbb{R}$ compact

Lemma 8.18 $\Rightarrow$ $\sup f(K)$, $\inf f(K)$ finite & belong to $f(K)$. So. f is bounded on K. and $\exists$ $p, q \in K$ s.t.

$$f(p) = \inf_K f(K), \quad f(q) = \sup f(K).$$

Done!

- Arzela-Ascoli is really about compactness

Recall

$$C^0[a,b] = \{f : [a,b] \to \mathbb{R} \mid f \text{ cont. on } [a,b]\}$$

Given f, g, define

$$d(f,g) := \sup_{x \in [a,b]} |f(x) - g(x)|$$

By EVT, $|f|, |g|$ is bounded on $[a,b]$. So.

$$|f(x) - g(x)| \leq \sup_{x \in [a,b]} |f(x)| + \sup_{x \in [a,b]} |g(x)|$$

& so $d(f,g) < \infty$.

Th^m 8.19 ① d is a metric on $C^0[a,b]$.

② $f_n \xrightarrow{d} f \iff f_n \xrightarrow{u.c} f$ on $[a,b]$.

③ $\mathcal{F} \subset C^0[a,b]$ is bounded $\iff$ $\mathcal{F}$ is a uniformly bounded family

Pf: Assignment

<u>Th^m 8.20</u> (Arzela - Ascoli, version - 2). A subset $\mathcal{F} \subset C^0[a,b]$ is compact $\Longleftrightarrow$ $\mathcal{F}$ is closed, bounded, and equicontinuous.

<u>Rk</u>: Equicont. is necessary, since consider

$$\mathcal{F} = \{ f : [0,1] \to \mathbb{R} \mid |f(x)| \le 1 \} \subset C^0[0,1].$$

Then $\mathcal{F}$ is closed & bounded. But $f_n(x) = x^n$ is a seq that has no convergent sub-seq^n in the metric d.

<u>Pf</u>: $\underline{\Longrightarrow}$ Sps $\mathcal{F}$ is compact. Then $\mathcal{F}$ is closed & bounded (true for any metric space). Sps $\mathcal{F}$ is not equicont. Then $\exists$ $\varepsilon > 0$ and sequences $f_n \in \mathcal{F}$, $x_n, y_n \in [a,b]$ s.t.

$$|x_n - y_n| < \frac{1}{n} \quad \underline{\text{but}} \quad |f_n(x_n) - f_n(y_n)| \geq \varepsilon.$$

$\mathcal{F}$ compact $\Longrightarrow$ $\exists$ sub-seq^n f_{n_k} s.t. $f_{n_k} \xrightarrow{u.c} f \in \mathcal{F}$. In particular, $\exists$ J_1 s.t. $\forall$ $k > J_1$, $\sup_{x \in [a,b]} |f_{n_k}(x) - f(x)| < \frac{\varepsilon}{3}$.

Also since $f \in C^0[a,b]$, $\exists$ $\delta > 0$ s.t.

$$|x - y| < \delta \Longrightarrow |f(x) - f(y)| < \varepsilon/3.$$

Let $k > J$ & s.t $1/n_k < \delta$. (so $|x_{n_k} - y_{n_k}| < \delta$).

Then

$$\varepsilon \le |f_{n_k}(x_{n_k}) - f(y_{n_k})| \le |f_{n_k}(x_{n_k}) - f(x_{n_k})| + |f(x_{n_k}) - f(y_{n_k})| + |f(y_{n_k}) - f_{n_k}(y_{n_k})|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Contradiction !

$\Longleftarrow$ Sps $\mathcal{F}$ is closed, bounded & equicont.

Let $\{f_n\}$ seq^n in $\mathcal{F}$. Then $\{f_n\}$ is uniformly bounded & equicont. So Arzela-Ascoli

$\Longrightarrow \exists$ sub seq f_{n_k} s.t $f_{n_k} \xrightarrow{u.c} f$.

Since each f_{n_k} is cont. $\Longrightarrow f \in C^0[a,b]$.

Also $f_{n_k} \xrightarrow{u.c} f \Longrightarrow f_{n_k} \xrightarrow{d} f$.

So f is a l.p of $\mathcal{F}$.

$\mathcal{F}$ closed $\Longrightarrow f \in \mathcal{F}$.

So any seq^n $\{f_n\}$ in $\mathcal{F}$ has a sub-seq $\{f_{n_k}\}$ converging to a point in $\mathcal{F}$.

So $\mathcal{F}$ compact.