WEEK - 6

Chapter 6 Sequences of functions

· Main problem

Def^n: Let $E \subseteq \mathbb{R}$ and $f_n : E \longrightarrow \mathbb{R}$ be a seq^n of functions. We say $\{f_n\}$ converges (pointwise) to f on E, and write $f_n \longrightarrow f$, if $\forall\, x \in E$

$$\lim_{n \to \infty} f_n(x) = f(x).$$

Or equivalently $\forall\, x \in E$, $\forall\, \varepsilon > 0$, $\exists\; N = N(\varepsilon, x)$ s.t

$$n > N \implies |f_n(x) - f(x)| < \varepsilon.$$

We then call f, the (pointwise) limit of $\{f_n\}$.

Ques Let $\{f_n\}$ a seq^n s.t $f_n \longrightarrow f$ on E.

1) If each f_n is cont, on E, is f cont on E. $\iff \forall\, p \in E$,

$$\lim_{x \to p} f(x) = f(p).$$

$$\iff \forall\, p \in E$$

$$\lim_{x \to p} \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} f_n(p)$$

$$= \lim_{n \to \infty} \lim_{x \to p} f_n(x)$$

i.e can you interchange limit?

2) Sps $E = [a,b]$ & $f_n \in R[a,b]$ $\forall n$. Is $f \in R[a,b]$? Moreover, is

$$\int_a^b f(t)\,dt = \int_a^b \lim_{n\to\infty} f_n(t)\,dt \overset{?}{=} \lim_{n\to\infty} \int_a^b f_n(t)\,dt.$$

3) Sps $E = (a,b)$ and each f_n is diff. on (a,b). Is f diff. on (a,b)? And is

$$f'(x) \overset{?}{=} \lim_{n\to\infty} f_n'(x)\ ?$$

Answer: No to all these!

Examples: 1) Let $f_n : [0,1] \longrightarrow \mathbb{R}$, $f_n(x) = x^n$.

Then f_n is cont. $\forall n$. For any $x \in [0,1)$, clearly $\lim\limits_{n\to\infty} x^n = 0$. On the other hand if $x = 1$, $x^n = 1$ $\forall n$, so $\lim\limits_{n\to\infty} x^n = 1$. So $f_n \longrightarrow f$ on $[0,1]$, where

$$f(x) = \begin{cases} 0, & x \in [0,1) \\ 1, & x = 1. \end{cases}$$

Clearly f is <u>dis cont.</u> at $x = 1$ even though f_n is cont on $[0,1]$.

2) $f_n : [0,1] \longrightarrow \mathbb{R}$ defined by

$$f_n(x) = n^2 x (1-x)^n.$$

<u>Claim</u>: $f_n \longrightarrow 0$ on $[0,1]$.

<u>Pf</u>: $f_n(0) = f_n(1) = 0 \ \forall n$. So w.l.o.g let $x \in (0,1)$. Then $(1-x^2) \in (0,1)$.

By L' Hospital's it follows that if $|a| < 1$ then

$$\lim_{n\to\infty} n^p a^n = 0$$

for any p. Applying this with $a = 1 - x^2$, clearly

$$\lim_{n\to\infty} f_n(x) = x \cdot \lim_{n\to\infty} n^2 (1-x^2)^n = 0.$$

So $f_n \longrightarrow 0$ on all of $[0,1]$.

Clearly the const. function $0 \in R[0,1]$ & $\int_0^1 0\, dt = 0$

But

$$\int_0^1 f_n(x)\, dx = n^2 \int_0^1 x(1-x^2)^n\, dx.$$

$$\underset{x\,dx = -\frac{du}{2}}{\overset{1-x^2=u}{=}} \frac{n^2}{2} \int_0^1 u^n\, du. = \frac{n^2}{2n+2} \xrightarrow{n\to\infty} \infty.$$

So

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx \neq \int_0^1 \lim_{n\to\infty} f_n(x)\, dx = 0.$$

3) For $n = 1, 2, \ldots$ let

$$f_n(x) = \lim_{m\to\infty} \cos(n!\,\pi x)^{2m}.$$

Note that

$$f_n(x) = \begin{cases} 1, & n!\,x \in \mathbb{Z} \\ 0, & \text{otherwise.} \end{cases}$$

<u>Claim</u>: $f_n \longrightarrow f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & \text{otherwise} \end{cases}$

<u>Pf</u>: Let $x \in \mathbb{Q}$, i.e $x = p/q$. Then for any $n \geq q$, $n!$ contains a factor of q, and so $n! \cdot p/q \in \mathbb{Z}$. In particular $\forall\, n \geq q$, $f_n(x) = 1$ and so $\lim_{n\to\infty} f_n(x) = 1$. On the other hand, if $x \in \mathbb{Q}^c$, then for any n, $n!\,x \notin \mathbb{Z}$, and so $f_n(x) = 0 \ \forall\, n$. In particular $\lim_{n\to\infty} f_n(x) = 0$.

<u>Claim</u>. f_n has only finitely many discont. on $[0,1]$ $\forall\, n$, and hence $f_n \in R[0,1]$ $\forall\, n$

<u>Pf</u>: If $x = p/q$ s.t $n!\,x \in \mathbb{Z}$, then

$$1 \leq q \leq n!$$

Moreover, since $p/q \in [0,1]$

$$0 \leq \frac{p}{q} \leq 1$$

So $$0 \le p \le n!$$

In particular there exists finitely many $x_1, \dots, x_{M_n}$ s.t $n!\,x_j \in \mathbb{Z}$ and $n!\,x \notin \mathbb{Z}$ if $x \in [0,1]$ & $x \neq x_j$.

Clearly f_n is only discont. at $x_1, \dots, x_{M_n}$.

So, $f_n \in R[0,1]$.

But $f_n \longrightarrow f$, where $f \notin R[0,1]$.

4) $f_n(x) = \dfrac{\sin(nx)}{\sqrt{n}}$ on $\mathbb{R}$. Clearly f_n is diff. on $\mathbb{R}$

& $f_n \longrightarrow f$, where $f(x) = 0 \ \forall x \in \mathbb{R}$.

Now, $f_n'(x) = \sqrt{n}\cos(nx)$. So $f_n'(0) \longrightarrow \infty$, but $f'(0) = 0$.

5) $f_n(x) = \sqrt{x^2 + \dfrac{1}{n}}$.

Clearly $f_n \longrightarrow f$ on $\mathbb{R}$, where $f(x) = \sqrt{x^2} = |x|$.

Each f_n is diff on $\mathbb{R}$, but f is not diff at 0.

Rk: The main problem is that pointwise convergence is not strong enough.

• Uniform convergence

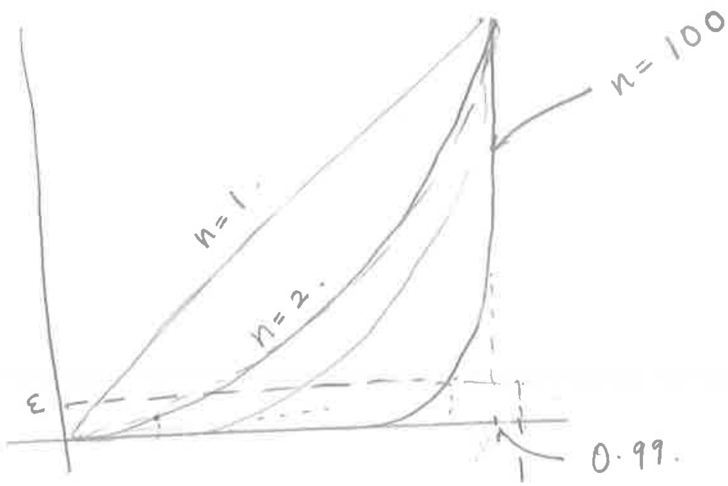
In pointwise convergence $f_n \to f$, $\forall\, x$, $\forall\, \varepsilon > 0$
$\exists\; N = N(x, \varepsilon)$ s.t

$$n > N \implies |f_n(x) - f(x)| < \varepsilon .$$

But N can grow bigger as 'x' varies

e.g. $f_n(x) = x^n$ on $[0,1]$.

$f_n \longrightarrow 0$ on $[0,1)$.



But

Given an x, to find N s.t $0 < f_n(x) < \varepsilon$ $\forall\; n > N$, as $x \longrightarrow 1$, $N \longrightarrow \infty$.

Def^n: Let $E \subseteq \mathbb{R}$, and $f_n : E \to \mathbb{R}$. We say $\{f_n\}$ <u>converges uniformly</u> to f on E, if $\forall\, \varepsilon > 0$, $\exists\; N = N(\varepsilon)$ s.t

$$\left.\begin{array}{l} n > N \\ x \in E \end{array}\right\} \implies |f_n(x) - f(x)| < \varepsilon .$$

We then write $f_n \xrightarrow{u.c} f$ on E

Rk 1) If $f_n \xrightarrow{u.c} f$, then in particular $f_n \to f$ pointwise. i.e $\lim_{n\to\infty} f_n(x) = f(x) \ \forall x \in E$

2) If $f_n \to f$ but $f_n \not\xrightarrow{u.c} f$, then $\exists\ \varepsilon > 0$ s.t $\forall N$, $\exists\ n > N$ & $x_n \in E$ s.t

$$|f_n(x_n) - f(x_n)| > \varepsilon.$$

Th^m 6.1 Sps $f_n \to f$. Then $f_n \not\xrightarrow{u.c} f$ on E $\iff \exists\ \varepsilon > 0$ and a seq^n $n_k \uparrow \infty$ and points $x_k \in E$ s.t

$$|f_{n_k}(x_k) - f(x_k)| > \varepsilon.$$

Pf $\Leftarrow$ Trivial.

$\Rightarrow$ Rk 2) shows that $\exists\ \varepsilon > 0$ s.t $\forall N$, $\exists\ n > N$ & $x_n \in E$ s.t $|f_n(x_n) - f(x_n)| \geq \varepsilon$.

Let $n_1 > 1$ s.t $\exists\ x_{n_1} = x_1$ s.t $|f_{n_1}(x_1) - f(x_1)| \geq \varepsilon$

Let $n_2 > n_1$ s.t $\exists\ x_2$ s.t $|f_{n_2}(x_2) - f(x_2)| \geq \varepsilon$.

Having picked $n_1, n_2, \dots, n_{k-1}$, let $n_k > n_{k-1}$ s.t $\exists\ x_k$ s.t $|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon$.

Then $\{n_k\}$ satisfies required properties

Rk: Most often Th^m is used in the foll form.

Cor 6.2: If $\exists\ \varepsilon > 0$ and a seq^n $\{x_n\}$ in E s.t $|f_n(x_n) - f(x_n)| \geq \varepsilon$, then $f_n \not\xrightarrow{u.c} f$

Ex: 1) Let $f_n(x) = \dfrac{1}{n(1+x^2)}$ on $\mathbb{R}$

Clearly $f_n \longrightarrow 0 \quad \forall x \in \mathbb{R}$.

Claim: $f_n \xrightarrow{u.c} 0$ on $\mathbb{R}$.

Pf: Let $\varepsilon > 0$. For any $x \in \mathbb{R}$

$$|f_n(x) - 0| = \frac{1}{n(1+x^2)} \leq \frac{1}{n}$$

Let $N > 1/\varepsilon$. Then $\forall n > N$, $\forall x \in \mathbb{R}$.

$$|f_n(x)| < \varepsilon$$

So $f_n \xrightarrow{u.c} 0$.

2) Let $f_n(x) = x^n$. Then $f_n \longrightarrow 0$ on $[0,1)$

Claim: $f_n \overset{u.c}{\not\longrightarrow} 0$ on $[0,1)$

Pf: Consider $x_n = 1 - 1/n$. Then

$$|f_n(x_n) - 0| = \left(1 - \frac{1}{n}\right)^n \xrightarrow{n\to\infty} e^{-1}.$$

Let $\varepsilon = e^{-1}/2$. Then $\exists$ N s.t $\forall n > N$.

$$\left(1 - \frac{1}{n}\right)^n > \varepsilon.$$

i.e $|f_n(x_n)| > \varepsilon$. So $f_n \overset{u.c}{\not\longrightarrow} 0$.

· Some sufficient conditions for uniform conv.

Def^n: A seq^n of functions $f_n : E \to \mathbb{R}$ is said to be uniformly Cauchy on E if $\forall\, \varepsilon > 0$

$\exists\, N$ s.t,

$$\left.\begin{array}{l} n, m > N \\ x \in E \end{array}\right\} \implies |f_n(x) - f_m(x)| \geq \varepsilon.$$

Th^m 6.3 Let $E \subseteq \mathbb{R}$ and $f_n : E \to \mathbb{R}$. TFAE

① $\{f_n\}$ converges uniformly.

② $\exists$ function $f : E \to \mathbb{R}$ s.t if $M_n = \sup\limits_{x \in E} |f_n(x) - f(x)|$

Then $\lim\limits_{n\to\infty} M_n = 0$

③ $\{f_n\}$ is uniformly Cauchy.

Pf (1) $\implies$ (2): Sps $f_n \xrightarrow{u.c} f$. Let $\varepsilon > 0$. Then $\exists$ N s.t $\forall\, n > N$, $\forall\, x \in E$

$$|f_n(x) - f(x)| < \varepsilon.$$

$$\implies \forall\, n > N,\ 0 \leq M_n = \sup_{x \in E} |f_n(x) - f(x)| \leq \varepsilon.$$

i.e $\forall\, \varepsilon > 0$, $\exists\, N$ s.t

$$n > N \implies 0 \leq M_n \leq \varepsilon$$

So $\lim\limits_{n\to\infty} M_n = 0$.

(2) $\Longrightarrow$ (3). Let $\varepsilon > 0$. Then $\exists$ N s.t

$$n > N \implies M_n < \frac{\varepsilon}{2}.$$

i.e $\forall x \in E$, $|f_n(x) - f(x)| < \frac{\varepsilon}{2}$. (*)

Now let $x \in E$ and $n, m > N$. Then (*) & triangle ineq $\Longrightarrow$

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So $\{f_n\}$ is uniformly Cauchy.

(3) $\Longrightarrow$ (1). Let $x \in E$. Since $\{f_n\}$ is uniformly Cauchy, in particular $\{f_n(x)\}$ is a Cauchy seq^n. So $\lim_{n \to \infty} f_n(x)$ exists. Let $f(x) = \lim_{n \to \infty} f_n(x)$.

i.e $f_n \longrightarrow f$ pointwise.

Claim $f_n \xrightarrow{u.c} f$ on E.

Pf: Let $\varepsilon > 0$. Then $\exists$ N s.t.

$$\left.\begin{array}{l} n, m > N \\ x \in E \end{array}\right\} \Longrightarrow |f_n(x) - f_m(x)| < \frac{\varepsilon}{2}. \quad (**)$$

Let $m \to \infty$. Since $|\cdot|$ is a cont. function

$$(**) \Longrightarrow |f_n(x) - f(x)| = \lim_{m \to \infty} |f_n(x) - f_m(x)|$$

$$\leq \frac{\varepsilon}{2} < \varepsilon.$$

i.e $\forall\ \varepsilon > 0,\ \exists\ N$ s.t

$$\left.\begin{array}{l} n > N \\ x \in E \end{array}\right\} \Longrightarrow |f_n(x) - f(x)| < \varepsilon.$$

So $f_n \xrightarrow{u.c} f$.

Rk (2) is sometimes useful to prove / disprove uniform conv.

e.g. Consider $f_n(x) = x^n$ on $[0,1)$. Then $f_n \longrightarrow 0$ on $[0,1)$. But

$$M_n = \sup_{[0,1)} |x^n| = 1$$

So $\lim_{n\to\infty} M_n \neq 0$. $(2) \Longrightarrow f_n \overset{u.c}{\not\longrightarrow} 0$.

· Uniform convergence and continuity

Thm 6.4 Let $f_n : E \longrightarrow \mathbb{R}$ be cont. at $p \in E$. If $f_n \xrightarrow{u.c} f$ on E, then f is cont at p.

Pf: ($\varepsilon/3$- trick). Let $p \in E$ and $\varepsilon > 0$.

Key observation For any x and n,

$$|f(x) - f(p)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(p)| + |f_n(p) - f(p)|.$$

$\exists\ N$ s.t $\forall\ t \in E$ (since $f_n \xrightarrow{u.c} f$).

$$|f_N(t) - f(t)| < \frac{\varepsilon}{3} \quad (*)$$

Also, f_N cont. $\Longrightarrow \exists\ \delta > 0$ s.t.

$$\left.\begin{array}{l} |x - p| < \delta \\ x \in E \end{array}\right\} \Longrightarrow |f_N(x) - f_N(p)| < \frac{\varepsilon}{3} \quad (**)$$

So for this $\delta > 0$, whenever $|x - p| < \delta$, $x \in E$,

$$|f(x) - f(p)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(p)| + |f_N(p) - f(p)|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

So f is cont. at p.

One can generalize this to a statement on interchanging limits

<u>Th^m 6.5</u>: Let $f_n : E \to \mathbb{R}$ and $f_n \xrightarrow{u.c} f$ on E.

Sps p is a l.p of E &

$$\lim_{x \to p} f_n(x) = A_n.$$

Then $\{A_n\}$ converges, $\lim_{x \to p} f(x)$ exists and

$$\lim_{x \to p} f(x) = \lim_{x \to p} \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \lim_{x \to p} f_n(x)$$

$$= \lim_{n \to \infty} A_n.$$

<u>Pf</u>: <u>Claim 1</u> $\{A_n\}$ is Cauchy.

Pf: Let $\varepsilon > 0$ $\exists$ N s.t.

$$\left.\begin{array}{l} n, m > N \\ x \in E \end{array}\right\} \implies |f_n(x) - f_m(x)| < \varepsilon/2.$$

$$\implies \lim_{x \to p} |f_n(x) - f_m(x)| \leq \varepsilon/2 < \varepsilon.$$

$$\implies |A_n - A_m| < \varepsilon.$$

So $\{A_n\}$ is Cauchy

Now Claim-1 $\implies$ $\{A_n\}$ converges. Let $A = \lim_{n \to \infty} A_n$.

<u>Claim-2</u> $\lim_{x \to p} f(x) = A$.

Pf: <u>Key obs</u>: $|f(x) - A| \leq |f(x) - f_N(x)| + |f_N(x) - A_N| + |A_N - A|$.

Since $f_n \xrightarrow{u.c} f$, $A_n \longrightarrow A$, $\exists$ N s.t.

(1) $\forall x \in E$, $|f(x) - f_N(x)| < \varepsilon/3$

(2) $|A_N - A| < \varepsilon/3$

Also, since $\lim_{x \to p} f_N(x) = A_N$, $\exists$ $\delta > 0$ s.t.

$$\left.\begin{array}{l} |x - p| < \delta \\ x \in E \end{array}\right\} \implies |f_N(x) - A_N| < \varepsilon/3.$$

But then, by the above observation,

$$\left.\begin{array}{l} |x - p| < \delta \\ x \in E \end{array}\right\} \implies |f(x) - A| < \varepsilon/3 + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

So $\lim_{x \to p} f(x) = A$.

- <u>Uniform Convergence and Integration</u>

<u>Th^m 6.6</u>: Let $f_n \in R[a,b]$ & $f_n \xrightarrow{u.c} f$ on $[a,b]$.
Then $f \in R[a,b]$. Moreover

$$\int_a^b f(t)\,dt = \lim_{n \to \infty} \int_a^b f_n(t)\,dt.$$

<u>Pf</u>: Let $M_n = \sup_{[a,b]} |f_n - f|$

Then $f_n \xrightarrow{u.c} f \implies \lim_{n \to \infty} M_n = 0$.

Now, $\forall\, t \in [a,b]$, $|f_n(t) - f(t)| \leq M_n$ i.e

$$f_n(t) - M_n \leq f(t) \leq f_n(t) + M_n.$$

$$\implies \quad (*) \begin{cases} U(f) \leq U(f_n + M_n) \\ L(f) \geq L(f_n - M_n) \end{cases}$$

Since $f_n \in R[a,b]$ & M_n is const, we have

$$U(f_n + M_n) = \int_a^b f_n(t)\,dt + M_n(b-a)$$

$$L(f_n - M_n) = \int_a^b f_n(t)\,dt - M_n(b-a).$$

Then $(*) \implies$

$$(**) \quad \int_a^b f_n(t)\,dt - M_n(b-a) \leq L(f) \leq U(f) \leq \int_a^b f_n(t)\,dt + M_n(b-a)$$

or

$$0 \leq U(f) - L(f) \leq 2M_n(b-a) \xrightarrow{n\to\infty} 0$$

So $U(f) = L(f)$ & hence $f \in R[a,b]$.

Also $(**) \implies$

$$\left| \int_a^b f(t)\,dt - \int_a^b f_n(t)\,dt \right| \leq M_n(b-a).$$

Given $\varepsilon > 0$, let N s.t $\forall\ n > N$, $M_n < \varepsilon/b-a$

Then $\forall\ n > N$,

$$\left| \int_a^b f(t)\,dt - \int_a^b f_n(t)\,dt \right| < \varepsilon.$$

So $\lim_{n\to\infty} \int_a^b f_n(t)\,dt = \int_a^b f(t)\,dt.$

Rk: The theorem does not generalize verbatim to improper integrals.

e.g. 1) Consider $f_n(x) = \begin{cases} 1/n, & 1 \leq x \leq n \\ 0, & x > n \end{cases}$

Then clearly $f_n \xrightarrow{u.c} 1/x$ on $[1,\infty)$. since

$$\sup_{x\in[1,\infty)} |f_n(x) - 1/x| = 1/n \longrightarrow 0.$$

But $1/x$ is not integrable on $[1,\infty)$ even though f_n is integrable on $[1,\infty)$ $\forall\, n$.

2) $g_n(x) = \begin{cases} 1/n, & 0 \leq x \leq n \\ 0, & x > n \end{cases}$

Now $g_n \xrightarrow{u.c} 0$ on $[0,\infty)$. But

$$\int_0^\infty g_n(x)\,dx = 1 \neq 0 = \int_0^\infty 0\,dx.$$

· <u>Uniform convergence and differentiation</u>

<u>Ques</u>: If $f_n \xrightarrow{u.c} f$ on (a,b) & f_n diff on (a,b), is f diff on (a,b).

<u>Ans</u>: NO! Consider $f_n : (-1,1) \to \mathbb{R}$

$$f_n(x) = \sqrt{x^2 + \frac{1}{n}}.$$

<u>Claim</u> $f_n \xrightarrow{u.c} |x|$ on $(-1,1)$

<u>Pf</u>:

$$\left| \sqrt{x^2 + \frac{1}{n}} - |x| \right| = \frac{\left(\sqrt{x^2 + 1/n} - |x|\right)\left(\sqrt{x^2+1/n} + |x|\right)}{\sqrt{x^2 + 1/n} + |x|}$$

$$= \frac{1}{n\left(\sqrt{x^2+1/n} + |x|\right)}$$

Now, $x^2, |x| \geq 0$. So $\sqrt{x^2+1/n} + |x| \geq \sqrt{1/n} = \frac{1}{\sqrt{n}}$.

So
$$\sup_{x \in (-1,1)} \left| \sqrt{x^2 + \frac{1}{n}} - |x| \right| \leq \frac{1}{n \cdot 1/\sqrt{n}} = \frac{1}{\sqrt{n}} \xrightarrow{n \to \infty} 0$$

Hence claim is proved.

Each f_n is diff on $(-1,1)$, but $|x|$ not diff at $x = 0$

<u>Th^m 6.7</u>. Sps $\{f_n\}$ is a seq^n of <u>diff.</u> functions on (a,b) s.t

(1) $f_n \longrightarrow f$ pointwise on (a,b)

(2) $f_n' \xrightarrow{u.c} g$ on (a,b)

Then $f_n \xrightarrow{u.c} f$ on (a,b), f is diff. on (a,b).

Moreover,

$$f'(x) = g(x) \quad \forall\ x \in (a,b).$$

<u>Rk</u>: If we assume that f_n' is cont on $[a,b]$. Then the theorem follows from Th^m 6.6 &

fundamental Th^m. Note that given $\varepsilon > 0$, since

$$f_n(x) = f_n(x_0) + \int_{x_0}^{x} f_n'(t)\,dt \quad , \text{ for some } x_0 \in (a,b)$$

Since $f_n' \xrightarrow{u.c} g$ and $f_n(x_0) \longrightarrow f(x_0)$ $\exists$ N s.t $\forall$ $n > N$, $\forall$ $t \in (a,b)$, (1) $|f_n'(t) - g(t)| < \varepsilon/2(b-a)$.

(2) $|f_n(x_0) - f(x_0)| < \varepsilon/2$.

If we let $F(x) = f(x_0) + \int_{x_0}^{x} g(t)\,dt$, one can see $\forall$ $x \in (a,b)$ & $n > N$,

$$|f_n(x) - F(x)| < \varepsilon$$

So $f_n \xrightarrow{u.c} F$. Since $f_n \longrightarrow f \Rightarrow F = f$ and f is diff with $f'(x) = g(x)$.

<u>Unfortunately</u> f_n' might not even be integrable.

<u>Pf of Th^m 6.7</u>: Let $c \in (a,b)$. Let φ be the difference quotients be

$$\varphi(t) = \frac{f(t) - f(c)}{t - c}, \quad \varphi_n(t) = \frac{f_n(t) - f_n(c)}{t - c}$$

<u>Claim 1</u> $f_n \xrightarrow{u.c} f$ on (a,b).

<u>Pf</u>: Let $\varepsilon > 0$ $\exists$ N s.t

(*) $\forall\ n, m > N$, $|f_n(t_0) - f_m(t_0)| < \varepsilon/2$. for some $t_0 \in (a,b)$ (since $\{f_n(t_0)\}$ converges).

(**) $\forall\ n, m > N$, $\forall\ x \in (a,b)$

$$|f_n'(x) - f_m'(x)| < \frac{\varepsilon}{2(b-a)}$$

since $f_n' \xrightarrow{u.c} g$.

Now, $|f_n(t) - f_m(t)| \le |f_n(t) - f_m(t) - [f_n(t_0) - f_m(t_0)]|$

$$+ |f_n(t_0) - f_m(t_0)|.$$

By MVT applied to $f_n - f_m$, $\exists\ x_0$ between t_0 & t s.t.

$$\frac{f_n(t) - f_m(t) - [f_n(t_0) - f_m(t_0)]}{t - t_0} = f_n'(x_0) - f_m'(x_0)$$

So if $n, m > N$

$$|f_n(t) - f_m(t) - [f_n(t_0) - f_m(t_0)]| \le |f_n'(x_0) - f_m'(x_0)|\,|t - t_0|$$

$$\overset{(**)}{\le} \frac{\varepsilon}{2(b-a)}(b-a) = \frac{\varepsilon}{2}.$$

So by (*) & (***), $\forall\ n, m > N$, $\forall\ t \in (a,b)$

$$|f_n(t) - f_m(t)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\Rightarrow$ $\{f_n\}$ is uniformly Cauchy.

So f_n converges uniformly. Since $f_n \to f$

$\Longrightarrow \quad f_n \xrightarrow{u.c} f$ on (a, b).

Claim 2 f is diff. $(\Longleftrightarrow \lim_{t \to c} \varphi(t)$ exists $)$ and

$$f'(c) = g(c). \quad (\Longleftrightarrow \lim_{t\to c} \varphi(t) = \lim_{n\to\infty} \lim_{t \to c} \varphi_n(t))$$

Pf: Note, f_n diff $\Longrightarrow \lim_{t \to x} \varphi_n(t) = f_n'(c)$.

$$t \neq c. \Longrightarrow \lim_{n\to\infty} \varphi_n(t) = \varphi(t)$$

Subclaim $\varphi_n \xrightarrow{u.c} \varphi$ on $(a,b) \setminus \{c\}$.

Assuming this, we apply Th^m 6.5 on interchanging limits with $A_n = f_n'(\ ) = \lim_{t\to x} \varphi_n(t)$.

Th^m 6.5 $\Longrightarrow \{A_n\}$ conv., $\lim_{t\to} \varphi(t)$ exists &

$$\lim_{t\to c} \varphi(t) = \lim_{n\to\infty} A_n = g(c).$$

So f is diff at c & $f'(c) = \lim_{t\to x} \varphi(t) = g(c)$

Pf of Subclaim Note

$$|\varphi_n(t) - \varphi_m(t)| = \frac{|f_n(t) - f_m(t) - [f_n(c) - f_m(c)]|}{|t-c|}$$

Again by MVT, $\exists\ t_0 \in (a,b)$ s.t.

$$\frac{|f_n(t) - f_m(t) - [f_n(c) - f_m(c)]|}{|t-c|} = |f_n'(t_0) - f_m'(t_0)|$$

Since $f_n' \xrightarrow{u\cdot c} g$, $\forall\ \varepsilon > 0$, $\exists\ N$ s.t $\forall\ x \in (a,b)$
$n, m > N \implies |f_n'(x) - f_m'(x)| < \varepsilon$.
So $\forall\ n, m > N$; applying this to $x = t_0$,

$$|\varphi_n(t) - \varphi_m(t)| < \varepsilon.$$

So $\{\varphi_n\}$ is uniformly Cauchy on $(a,b)\setminus\{c\}$.
& hence uniformly convergent.
Since $\varphi_n \longrightarrow \varphi$ pointwise $\implies \varphi_n \xrightarrow{u\cdot c} \varphi$ on $(a,b)\setminus\{c\}$.

- Series of functions

Def^n Given a seq^n $f_n : E \to \mathbb{R}$, $n = 0, 1, \dots$, we denote the seq^n of partial sums $s_n : E \to \mathbb{R}$ by

$$s_n(x) = \sum_{k=0}^{n} f_k(x).$$

We say $\sum f_n$ converges uniformly (resp. pointwise) to f if $s_n \xrightarrow{u\cdot c} f$ (resp. $s_n \longrightarrow f$). We then denote

$$\sum_{n=0}^{\infty} f_n(x) = \lim_{n\to\infty} s_n(x).$$

Th^m 6.8 (Cauchy criteria) $\sum f_n$ conv. uniformly
$\iff \forall\ \varepsilon$, $\exists\ N$ s.t $\forall\ n \geq m > N$, $\forall\ x \in E$,

$$\left|\sum_{k=m}^{n} f_k(x)\right| < \varepsilon.$$

Pf: Apply uniform Cauchy criteria to $\{S_n\}$.

Th^m 6.9 (Weierstrass M-test). Sps $f_n : E \to \mathbb{R}$ & M_n s.t.

$$|f_n(x)| \leq M_n \quad \forall n, \ \forall x \in E.$$

Then $\sum M_n$ conv. $\Longrightarrow$ $\sum f_n$ conv. uniformly.

Pf. Let $\varepsilon > 0$. $\sum M_n$ conv $\Longrightarrow$ $\exists N$ s.t. $\forall n \geq m > N$,

$$\sum_{k=m}^{n} M_k < \varepsilon.$$

Δ-ineq $\Longrightarrow$ $\forall n \geq m > N$ and $\forall x \in E$,

$$\left|\sum_{k=m}^{n} f_k(x)\right| \leq \sum_{k=m}^{n} |f_k(x)|$$

$$< \sum_{k=m}^{n} M_k < \varepsilon.$$

So $\sum f_n$ satisfies uniform Cauchy for series

So Th^m 6.8 $\Longrightarrow$ $\sum f_n$ conv. uniformly.

Example: (Fourier series). Consider $\sum a_n \sin(nx)$.
Since $|a_n \sin(nx)| \leq |a_n|$. Th^m 6.9 says that $\sum |a_n|$ conv. $\Longrightarrow$ $\sum a_n \sin(nx)$ converges uniformly.

Th^m 6.10 ① If $f_n : E \longrightarrow \mathbb{R}$ is cont and $\sum f_n$ conv. uniformly, then $\sum f_n$ is cont.

② If $f_n \in R[a,b]$ $\forall n$, and $\sum f_n$ conv. uniformly then $f = \sum f_n \in R[a,b]$. Moreover

$$\int_a^b f(t)\,dt = \sum_{n=0}^{\infty} \int_a^b f_n(t)\,dt.$$

③ If $f_n : (a,b) \longrightarrow \mathbb{R}$ diff s.t $\sum f_n$ conv. pointwise, & $\sum f_n'$ conv uniformly on (a,b) & Then $\sum f_n$ conv. uniformly &

$$\frac{d}{dt} \sum_{n=0}^{\infty} f_n(t) = \sum_{n=0}^{\infty} f_n'(t)$$

Pf (2) Let $s_n = \sum_{k=0}^{n} f_k$. Then $s_n \in R[a,b]$ &

$$\int_a^b s_n(t)\,dt = \int_a^b \sum_{k=0}^{n} f_k(t)\,dt$$

$$= \sum_{k=0}^{n} \int_a^b f_k(t)\,dt.$$

Since $s_n \xrightarrow{u.c} f$, $f \in R[a,b]$ &

$$\int_a^b f(t)\,dt = \lim_{n\to\infty} \int_a^b s_n(t)\,dt$$
$$= \lim_{n\to\infty} \sum_{k=0}^{n} \int_a^b f_k(t)\,dt$$
$$= \sum_{k=0}^{\infty} \int_a^b f_k(t)\,dt.$$

(1) & (3) are similar

• Power Series

Def^n: Given a seq^n $\{C_n\}$ in $\mathbb{R}$, the series

$$\sum_{n=0}^{\infty} C_n (x-a)^n$$

is called a power series centered at 'a'. The number C_n is called the n^{th} coefficient of the series

Ques: For what values of x does the series converge? Converge uniformly? What properties does the resulting function have?

Ex. $C_n = 1$, $a = 0$. Then $\sum x^n$ converges $\forall$ $|x|<1$

Weierstrass M-test $\Rightarrow$ $\sum x^n$ converges uniformly to $\frac{1}{1-x}$ on $[-r, r]$ $\forall$ $r < 1$

Th^m 6.11 (Fundamental Theorem of power series)

Consider the power series $\sum c_n(x-a)^n$

1) The series converges absolutely on $|x-a|<R$ & diverges on $|x-a|>R$, where

$$R = \frac{1}{\limsup\limits_{n\to\infty} |c_n|^{1/n}} \in [0,\infty].$$

R is called the <u>radius of convergence</u>.

2) Convergence is <u>uniform</u> on $|x-a|\le r$ $\forall$ $r<R$.

3) For $x\in(a-R,\ a+R)$, if we write

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n,$$

Then f is diff on $(a-R,\ a+R)$ and

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^n$$

4) Also, $f\in R_{loc}(a-R,\ a+R)$ and for any $c,d\in(a-R,\ a+R)$,

$$\int_c^d f(x)\,dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}\Bigg|_c^d .$$

<u>Rk</u>: The above theorem does <u>not</u> tell us anything about conv at $x=a-R$ or $a+R$.

In any case, the set

$$I := \{x \mid \sum c_n (x-a)^n \text{ conv}\}$$

is an interval (open, closed or half open / closed) & is called the <u>interval</u> of <u>convergence</u>.

<u>Pf of Th^m 6.11</u> W.l.o.g suppose $a = 0$

(else simply change coordinates $t = x - a$).

So our series is $\sum_{n=0}^{\infty} c_n x^n$

1) Let $a_n = c_n x^n$. Then

$$\limsup_{n\to\infty} |a_n|^{1/n} = \limsup_{n\to\infty} |c_n|^{1/n} |x^n|^{1/n}$$

$$= |x| \limsup_{n\to\infty} |c_n|^{1/n}$$

$$= \frac{|x|}{R}.$$

By root test,

$\sum a_n$ converges absolutely if $\frac{|x|}{R} < 1$ or $|x| < R$.

$\sum a_n$ diverges if $\frac{|x|}{R} > 1$ or $|x| > R$.

2) Let $r < R$. By previous part $\sum c_n r^n$ converges. Also $\forall\ x$ s.t $|x| \le r$,

$$|c_n x^n| < c_n r^n .$$

Weierstrass M-test (with $M_n = c_n r^n$) $\Longrightarrow$

$\sum c_n x^n$ conv. uniformly on $|x| \le r$.

③ Apply theorem on uniform conv. & diff.

Let $|x| < r < R$, and

$$s_n(x) = \sum_{k=0}^{n} c_k x^k$$

Claim (a) $s_n \xrightarrow{u.c} f$ on $|x| < r$.

(b) $\{s_n'\}$ converges uniformly on $|x| < r$.

Pf (a) follows from ① & ② above since $r < R$.

Now,
$$s_n' = \sum_{k=1}^{n} k \cdot c_k x^k$$

Let $a_n = n c_n x^n$ and $g = \sum_{n=1}^{\infty} n \cdot c_n x^{n-1}$

$$\limsup_{n \to \infty} |a_n|^{1/n} = |x| \limsup_{n\to\infty} n^{1/n} |c_n|^{1/n}$$

$$= \frac{|x|}{R} \quad (\text{since } \lim_{n\to\infty} n^{1/n} = 1).$$

$$< 1 \quad \text{since } |x| < r < R$$

So g converges uniformly i.e

$$s_n' \xrightarrow{u.c} g \text{ on } (-r, r).$$

Then by Th^m 6.7 f is diff on $(-r, r)$ &

$$f'(x) = \lim_{n\to\infty} s_n'(x)$$

$$= g(x)$$

$$= \sum_{n=1}^{\infty} n c_n x^{n-1}.$$

4) Again, let $s_n(x) = \sum_{k=0}^{n} c_k x^k$. Let $c, d \in (-r, r)$ for $r < R$.

Then $s_n \in R[c, d]$ &

$$\int_c^d s_n(x)\,dx = \sum_{k=0}^{n} c_k \frac{x^{k+1}}{k+1}\Big|_c^d$$

Since $s_n \xrightarrow{u.c} f$ on $(-r, r)$, $f \in R[c, d]$.

&

$$\int_c^d f(x)\,dx = \sum_{k=0}^{\infty} c_k \frac{x^{k+1}}{k+1}\Big|_c^d$$

<u>Examples</u> ① $\sum x^n$ Then $c_n = 1$, $a = 0$.

$\limsup_{n\to\infty} |c_n|^{1/n} = 1$, so $R = 1$.

$$\sum x^n \xrightarrow{u.c} \frac{1}{1-x} \text{ on } |x| < r \ \forall\, r < 1$$

Th^m $\Longrightarrow$ Series conv. on $(-1,1)$.

Boundary points $x = 1$. Series is $\sum 1$ and so diverges

$x = -1$, series is $\sum(-1)^n$ and so diverges

So $I = (-1, 1)$.

2) $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-1)^n}{n} = \sum_{n=1}^{\infty} \frac{2^n}{n}(x - 1/2)^n$, Here $C_n = 2^n/n$

$$\limsup_{n\to\infty} |C_n|^{1/n} = 2 \cdot \limsup_{n\to\infty} \frac{1}{n^{1/n}} = 2$$

So $R = 1/2$. So series conv. on $|x - 1/2| < 1/2$ or $\forall\ x \in (0, 1)$.

Boundary points $x = 0$. The series becomes $\displaystyle\sum_{n=1}^{\infty}(-1)^n/n$ which converges

$x = 1$ Series becomes $\sum 1/n$ which diverges

So $I = [0, 1)$.

3) (More series from geometric).

We have seen $\sum x^n \xrightarrow{u.c.} \dfrac{1}{1-x}$ on $(-r, r)$ $\forall\ r < 1$. Differentiating

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + \cdots$$

Similarly
$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

So $\forall \ x \in (-1,1)$,
$$\ln(1+x) = \int_0^x \frac{dt}{1+t} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \cdot x^{n+1}$$
$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

<u>Cor</u> 6.12 If $\sum_{n=0}^{\infty} c_n (x-a)^n$ is a power series s.t $\lim_{n\to\infty} |c_{n+1}/c_n|$ exists. Then
$$R = \frac{1}{\lim_{n\to\infty} |c_{n+1}/c_n|}.$$

<u>Pf</u>: In assignment problem we saw that if $\lim_{n\to\infty} |c_{n+1}/c_n|$ exist, then $\lim_{n\to\infty} |c_n|^{1/n}$ exist

&
$$\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty} |c_n|^{1/n} = \frac{1}{R}.$$

<u>Examples</u> 1) (Exponential). Consider $\sum_{n=0}^{\infty} x^n/n!$

Then $c_n = 1/n!$
$$\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

So $R = \infty$. i.e $\sum \frac{x^n}{n!}$ conv. on $(-\infty, \infty)$.

Def^n: We define the exponential function by $\exp : \mathbb{R} \to \mathbb{R}$

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \text{Note } \begin{array}{l} \exp(1) = e \\ \exp(0) = 1. \end{array}$$

Th^m 6.11 $\Longrightarrow$ $\exp(x)$ is diff on $\mathbb{R}$. Moreover

$$\frac{d}{dx}\exp(x) = \sum_{n=1}^{\infty} \frac{n \cdot x^{n-1}}{n!}$$

$$= \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$$

$$\overset{n-1=m}{=\!=} \sum_{m=0}^{\infty} \frac{x^m}{m!} = \exp(x).$$

2) (Trig functions). We define $\sin, \cos : \mathbb{R} \to \mathbb{R}$.

$$\sin(x) := \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!}$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!}$$

Check: Both series converge on $(-\infty, \infty)$.

Th^m 6.11 $\implies$ $\sin(x)$, $\cos(x)$ diff on $\mathbb{R}$

$$\frac{d}{dx}\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)\, x^{2n}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$= \cos(x).$$

Similarly $\frac{d}{dx}\cos x = -\sin x$.

· <u>Taylor series of a function (revisited)</u>

<u>Cor 6.13</u> Let $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ s.t $R > 0$.

Then ① f has a derivative of order 'n' for all n. Moreover

$$f^{(n)}(x) = \sum_{k=n}^{\infty} c_k\, k(k-1)\cdots(k-n+1)(x-a)^{k-n}$$

② In particular, $\boxed{c_n = \frac{f^{(n)}(a)}{n!}}$

<u>Pf</u>: (1) Pf by induction

<u>Base Case</u> $n=1$. By Th^m 6.11

<u>Inductive Step</u>: Sps Th^m proved for $n=1,2,\dots,m$

Then $f^{(m)}(x) = \sum_{k=m}^{\infty} C_k k \cdots (k-m+1)(x-a)^{k-m}$

One can check that power series for $f^{(m)}(x)$ also has radius of conv. $= R$. So again Th^m 6.11 $\Rightarrow$ $f^{(m)}$ is diff on $(a-R, a+R)$ &

$$f^{(m+1)}(x) = \frac{d}{dx} f^{(m)}(x)$$

$$= \sum_{k=m+1}^{\infty} C_k k \cdots (k-m+1)(k-m)(x-a)^{k-m-1}$$

$$= \sum_{k=m+1}^{\infty} C_k k \cdots (k-(m+1)+1)(x-a)^{k-(m+1)}$$

So statement also true for $n = m+1$.

(2) Clearly

$$f^{(n)}(a) = C_n \cdot n(n-1) \cdots (n-n+1)$$

$$= C_n \cdot n!$$

Done!

Rk: This implies if $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$, then Taylor series $T_f(a;x) = f(x)$.

Ques: Can all functions f with derivatives of all orders, be represented by their Taylor series?

<u>Ex</u>: Answer in general is no!

Consider

$$f(x) = \begin{cases} \exp\left(\frac{-1}{x^2}\right), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

<u>Claim</u>: f is diff on $\mathbb{R}$ & $f'(0) = 0$.

<u>Pf</u>: Diff quotient is

$$\varphi(x) = \frac{f(x) - f(0)}{x - 0} = \frac{1}{x} \exp\left(-\frac{1}{x^2}\right).$$

Then $\varphi(0+) = \lim_{x \to 0^+} \varphi(x) \overset{1/x = t}{=} \lim_{t \to \infty} t \exp(+t^2).$

$$= \lim_{t \to \infty} \frac{t}{\exp(t^2)}$$

$$\overset{\text{L'Hospital}}{=} \lim_{t \to \infty} \frac{1}{2t \exp(t^2)} = 0.$$

Similarly $\varphi(0-) = 0$.

So $f'(0)$ exist & $f'(0) = 0$.

<u>Claim</u> $f^{(n)}(x)$ exists $\forall\, x \in \mathbb{R}$ & $n = 1, 2, \ldots$

Moreover $f^{(n)}(0) = 0$.

<u>Pf</u>: Assignment.

So the Taylor series for f is

$$T(0,x) = f(0) + f'(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \cdots$$

$$= 0$$

but $f(x) \neq 0 \quad \forall\, x \neq 0$.

So in this case $f(x) = T(0,x)$ only at $x = 0$.

Rk: Given, f, the only way to know if $f(x) = T_f(a;x)$ on $|x-a| < r$ is to show that $R_n(a;x) = f(x) - T(a;x) \xrightarrow{n\to\infty} 0$ on $|x-a| < r$ using Taylor's est.

In the example above for any $x \neq 0$,

$$\frac{f^{(n+1)}(x)}{(n+1)!} \xrightarrow{n\to\infty} \infty$$

So $R_n \not\to 0$

· The exponential function

Recall.

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad \begin{aligned} \exp(0) &= 1 \\ \exp(1) &= e. \end{aligned}$$

$$\frac{d}{dx}\exp(x) = \exp(x).$$

Thm 6.14 1) $\exp(-x) = \dfrac{1}{\exp(x)}$

2) $A\exp(x)$ is the unique function $f: \mathbb{R} \to \mathbb{R}$

s.t

$$\begin{cases} f' = f \\ f(0) = A \end{cases}$$

3) $\exp(x+y) = \exp(x)\exp(y)$

4) $\exp(x) = e^x$

5) $e^x \neq 0 \ \forall \ x \in \mathbb{R}$, and hence $e^x > 0 \ \forall \ x$.

6) strictly mononotonically increasing

and $$\lim_{x \to \infty} e^x = \infty$$

Pf: 1) $\exp(x) \cdot \exp(-x) = \left(1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} \cdots\right)$

$$\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right)$$

$$= 1 + x - x + \frac{x^2}{2!} + \frac{x^2}{2!} - x^2 + \cdots$$

In general,

$$\exp(x)\exp(-x) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \frac{(-1)^{n-k}}{k!(n-k)!}\right) x^n$$

$$= 1 + \sum_{n=1}^{\infty}\left(\sum_{k=0}^{n} \frac{(-1)^{n-k}}{k!(n-k)!} x^n\right)$$

$$= 1 + \sum_{n=\phi}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \right) \frac{x^n}{n!}$$

$$= 1 + \sum_{n=1}^{\infty} (1-1)^n \frac{x^n}{n!} \quad \text{(by binomial theorem)}$$

$$= 1.$$

2) Sps f is any function solving $f' = f$ & $f(0) = A$. Consider $g(x) = f(x)\exp(-x)$

Then

$$g'(x) = f'\exp(-x) - f(x)\exp(-x)$$

$$= (f' - f)\exp(-x) = 0.$$

So $g' \equiv 0$ on $\mathbb{R}$, Hence $g(x) = g(0)\ \forall x$.

But $g(0) = f(0)\exp(0) = A$.

So $f(x)\exp(-x) = A \quad \forall x.$

1) $\Rightarrow$ $f(x) = A\exp(x) \quad \forall x \in \mathbb{R}.$

3) For a fixed $y \in \mathbb{R}$, define

$$f_y(x) = \exp(x+y)$$

Then chain rule $\Rightarrow$ $f_y'(x) = f_y(x)$.

So 2) $\Rightarrow$ $f_y(x) = f_y(0)\cdot\exp(x) = \exp(y)\exp(x)$.

$$\Rightarrow \exp(x+y) = \exp(x)\exp(y).$$

4) 3) $\Longrightarrow$ $\forall\ x \in \mathbb{R},\ \forall\ n \in \mathbb{N}$

$$\exp(nx) = [\exp(x)]^n$$

Claim $\exp\left(\frac{p}{q}\right) = [\exp(1)]^{p/q} = e^{p/q}$

Pf: By obs above if $x = p/q$ and $n = q$, then

$$\exp(p) = \left[\exp\left(\frac{p}{q}\right)\right]^q$$

Also $\exp(p) = [\exp(1)]^p$

So $\left[\exp\left(\frac{p}{q}\right)\right]^q = \exp(1)^p$

Taking q^{th} root we are done. Since $\mathbb{Q}^c$ is dense in $\mathbb{R}$ & exp & e^x are cont $\Rightarrow$ $\exp(x) = e^x\ \forall\ x \in \mathbb{R}$

5) If $\exp(x) = 0$, then $\exp(x) \cdot \exp(-x) = 1$

Contradiction. Since $\exp(1) = e > 0$,

IVT $\Longrightarrow$ $\exp(x) > 0 \quad \forall\ x$

6). $\frac{d}{dx}\exp(x) = \exp(x) > 0 \Longrightarrow \exp(x) \uparrow_{\text{strict}}$

Since $\exp(x) = e^x$ and $e > 1$.

clearly $\lim_{x \to \infty} \exp(x) = \infty$

- **Trigonometric functions**

Recall that $\sin(x), \cos(x) : \mathbb{R} \longrightarrow \mathbb{R}$ are defined by

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots$$

We have already seen that

$$\frac{d \sin x}{dx} = \cos x, \quad \frac{d \cos x}{dx} = -\sin x.$$

$\underline{\text{Th}^m}$ 1) $\sin(-x) = -\sin(x)$, $\cos(-x) = \cos(x)$, $\sin(0) = 0$, $\cos(0) = 1$

2) $\sin(x+y) = \sin x \cos y + \cos x \sin y$.

$\cos(x+y) = \cos x \cos y - \sin x \sin y$.

3) $\sin^2 x + \cos^2 x = 1$.

Pf: 1) is trivial from the definition

2) For this we use the binomial theorem, that

$$(x+y)^m = \sum_{k=0}^{m} \binom{m}{k} \cdot x^k y^{m-k}$$

So,

$$\sin(x+y) = \sum_{n=0}^{\infty} (-1)^n \frac{(x+y)^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \sum_{k=0}^{2n+1} \binom{2n+1}{k} x^k \, y^{2n+1-k}.$$

Recall that $\binom{2n+1}{k} = \dfrac{(2n+1)!}{k!\,(2n+1-k)!}$, so

$$\sin(x+y) = \sum_{n=0}^{\infty} (-1)^n \sum_{k=0}^{2n+1} \frac{x^k}{k!} \frac{y^{2n+1-k}}{(2n+1-k)!}$$

We split the inside sum into case when k is odd & k is even i.e when $k = 2j+1$. or $k = 2j$, $j = 0, 1, \ldots, n$. That is,

$$\sum_{k=0}^{2n+1} \frac{x^k \, y^{2n+1-k}}{k!(2n+1-k)!} = \sum_{j=0}^{n} \frac{x^{2j+1}}{(2j+1)!} \cdot \frac{y^{2(n-j)}}{[2(n-j)]!}$$

$$+ \sum_{j=0}^{n} \frac{x^{2j}}{(2j)!} \frac{y^{2(n-j)+1}}{[2(n-j)+1]!}$$

So

$$\sin(x+y) = \sum_{n=0}^{\infty} (-1)^n \sum_{j=0}^{n} \frac{x^{2j+1}}{(2j+1)!} \frac{y^{2(n-j)}}{[2(n-j)]!} \qquad (*)$$

$$+ \sum_{n=0}^{\infty} (-1)^n \sum_{j=0}^{n} \frac{x^{2j}}{(2j)!} \frac{y^{2(n-j)+1}}{[2(n-j)]!}$$

Now,

$$\sin x \cos y = \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j+1}}{(2j+1)!} \cdot \sum_{l=0}^{\infty} (-1)^l \frac{y^{2l}}{(2l)!}$$

We use the distributive property to multiply, and focus on the deg. $2n+1$ terms i.e terms of the form $x^{2j+1} y^{2l}$ s.t $2j+1+2l = 2n+1$ i.e $l = n-j$, $j = 0, 1, \dots, n$.
Then $j + l = n$. so $(-1)^j(-1)^l = (-1)^n$. So we write

$$\sin(x)\cos y = \sum_{n=0}^{\infty} (-1)^n \sum_{j=0}^{n} \frac{x^{2j+1}}{(2j+1)!} \frac{y^{2(n-j)}}{[2(n-j)]!}$$

Similarly

$$\cos x \sin y = \sum_{n=0}^{\infty} (-1)^n \sum_{j=0}^{n} \frac{x^{2j}}{[2j]!} \frac{y^{2(n-j)+1}}{[2(n-j)+1]!}$$

Then $(*) \Longrightarrow \sin(x+y) = \sin x \cos y + \cos x \sin y$

Similarly, one can show.

$$\cos(x+y) = \cos x \cos y - \sin x \sin y.$$

3) Let $x = -y$ in the $\cos(x+y)$ formula.

Then

$$1 = \cos(0) = \cos(x)\cos(-x) - \sin(x)\sin(-x)$$
$$= \cos^2 x + \sin^2 x.$$

In fact sine & cosine turn out to be periodic. i.e

<u>Th^m</u>: There exists a real number, denoted by π, which is greater than zero & s.t $\sin(x+2\pi) = \sin(x)$ $\forall\, x \in \mathbb{R}$. Moreover, if $\beta \in \mathbb{R}$ s.t $\sin(x+\beta) = \sin(x)$, then $\beta = 2n\pi$ for some $n \in \mathbb{Z}$.

Similarly $\cos(x+2\pi) = \cos(x)$ & 2π is essentially the unique period for cos.

<u>Rk</u> The proof, starting with just the power series is non trivial! See the addendum to the Assignment - 6.