WEEK - 5

· Criteria for integrability

The main theorem we want to prove is the foll

Thm 5.1 Let $f : [a,b] \longrightarrow \mathbb{R}$ be bounded. TFAE

1) $f \in \mathcal{R}[a,b]$
2) $f \in \mathcal{D}[a,b]$
3) $\forall \varepsilon > 0$, $\exists$ partition P s.t
$$U(P,f) - L(P,f) < \varepsilon$$
4) $\forall \varepsilon > 0$, $\exists$ $\delta > 0$ s.t
$$|P| < \delta \implies U(P,f) - L(P,f) < \varepsilon.$$

Before we can prove this, we need some preparation

Defn: Given a partition P, another partition P^* is called a <u>refinement</u> of P if $P \subseteq P^*$

Ex Consider $[0,1]$. $P = \{0, 1/4, 1/2, 3/4, 1\}$.
Then $P_1^* = \{0, 1/6, 1/3, 1/2, 2/3, 5/6, 1\}$ is <u>NOT</u> a refinement, while $P_2^* = \{0, 1/6, 1/4, 1/2, 2/3, 3/4, 1\}$ is a refinement.

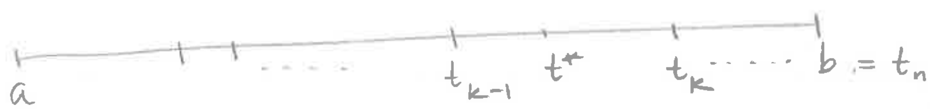
Lemma 5.2 If $\mathcal{P} \subseteq \mathcal{P}^*$, then

$$L(\mathcal{P}, f) \underset{(1)}{\leq} L(\mathcal{P}^*, f) \leq U(\mathcal{P}^*, f) \underset{(2)}{\leq} U(\mathcal{P}, f).$$

Pf: We prove (1). Pf of (2) is similar.

Let $\mathcal{P} = \{t_0, t_1, \ldots, t_n\}$. Sps $\mathcal{P}^*$ has one extra point. If $\mathcal{P}^*$ has more extra points, simply repeat the argument below.

Let $t^* \in (t_{k-1}, t_k)$ and $\mathcal{P}^* = \{t_0, t_1, \ldots, t_{k-1}, t^*, t_k, \ldots t_n\}$



As usual, let $m_j = \inf\limits_{[t_{j-1}, t_j]} f(t)$.

Also, let $W_1 = \inf\limits_{[t_{k-1}, t^*]} f(t)$, $W_2 = \inf\limits_{[t^*, t_k]} f(t)$

Key point: $W_1, W_2 \geq m_k$.

Now,

$$L(\mathcal{P}^*, f) = \sum_{j=1}^{k-1} m_j \Delta t_j + W_1(t^* - t_{k-1}) + W_2(t_k - t^*) + \sum_{j=k+1}^{n} m_j \Delta t_j.$$

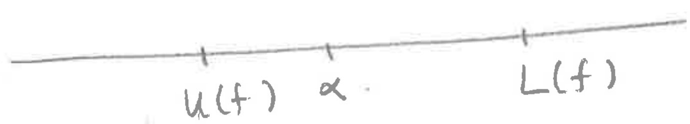
$$\geq \sum_{j=1}^{k-1} m_j \,\Delta t_j + m_k (t^* - t_{k-1}) + m_k (t_k - t^*) + \sum_{j=k+1}^{n} m_j \,\Delta t_j$$

$$= L(P, f).$$

Cor 5.3 For any $f : [a,b] \longrightarrow \mathbb{R}$,

$$L(f) \leq U(f).$$

Pf: Sps not. That is, sps $U(f) < L(f)$.
Let α s.t. $U(f) < \alpha < L(f)$.



Then $\exists$ parti~~titions~~ P_1 and P_2 s.t.

$$U(P_1, f) < \alpha, \quad L(P_2, f) > \alpha.$$

Let $P = P_1 \cup P_2$. Then P is a refinement of both P_1 and P_2. So, by Lemma 5.2

$$\alpha < L(P_2, f) \leq L(P, f) \leq U(P, f) \leq U(P_1, f) < \alpha.$$

Contradiction!

Pf of Thm: We prove (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2): Sps $f \in \mathcal{R}[a,b]$. Let $\varepsilon > 0$. $\exists\ \delta > 0$

s.t

$$|P| < \delta \implies |S(f, P, \{t_k^*\}) - r| < \varepsilon, \quad (*)$$

where $r = \mathcal{R}\int_a^b f(t)\,dt$, and $t_k^* \in (t_{k-1}, t_k)$.

Claim: $f \in \mathcal{D}[a,b]$ & $\int_a^b f(t)\,dt = r$.

Pf: Let P a partition with $|P| < \delta$. Say $P = \{t_0, t_1, \dots, t_n\}$ where $\Delta t_k = t_k - t_{k-1} = (b-a)/n$

& n large enough s.t $(b-a) < n\delta$.

Let $t_k^* \in (t_{k-1}, t_k)$ s.t

$$f(t_k^*) \le m_k + \varepsilon.$$

Then

$$S(f, P, \{t_k^*\}) = \sum_{k=1}^n f(t_k^*)\,\Delta t_k$$

$$\le \sum_{k=1}^n m_k\,\Delta t_k + \varepsilon \underbrace{\sum_{k=1}^n \Delta t_k}_{= b-a}$$

$$= L(P, f) + \varepsilon(b-a).$$

Also, $(*) \implies S(f, P, \{t_k^*\}) \ge r - \varepsilon,$

and so

$$L(P, f) \ge r - \varepsilon - \varepsilon(b-a).$$

Since $L(f) \geq L(P, f)$

$\Longrightarrow \quad L(f) \geq r - \varepsilon - \varepsilon(b-a) \quad \forall \; \varepsilon > 0.$

So, $\boxed{L(f) \geq r}$

Similarly, one can show $U(f) \leq r$.

Cor 5.3 $\Longrightarrow$ $U(f) = L(f) = r$.

So $f \in \mathcal{D}[a, b]$ and $\int_a^b f(t)\,dt = r$.

(2) $\Longrightarrow$ (3): Sps $f \in \mathcal{D}[a,b]$. Then

$$\sup_P L(P, f) = \inf_P U(P, f) = A = \int_a^b f(t)\,dt$$

Let $\varepsilon > 0$. $\exists$ P_1 s.t

$$U(P_1, f) < A + \frac{\varepsilon}{2} \qquad (**)$$

$$L(P_2, f) > A - \frac{\varepsilon}{2}.$$

Let $P = P_1 \cup P_2$. Then P is a refinement, and so Lemma 5.2 $\Longrightarrow$ $U(P, f) \leq U(P_1, f)$ and $L(P, f) \geq L(P_2, f)$. So $(**)$ $\Longrightarrow$

$$U(P, f) < A + \frac{\varepsilon}{2}$$

$$L(P, f) > A - \frac{\varepsilon}{2}.$$

Subtracting

$$U(P, f) - L(P, f) > \varepsilon.$$

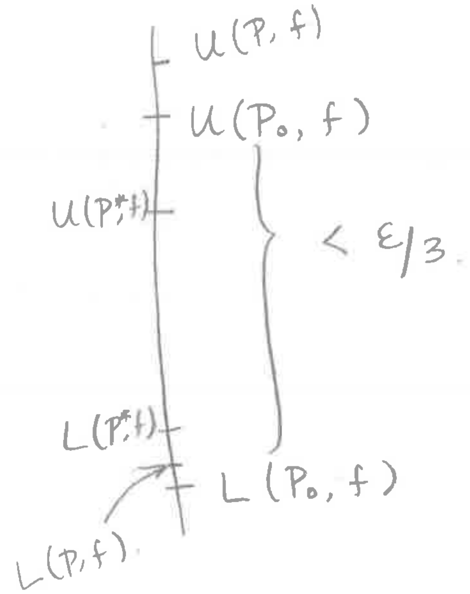
(3) $\Longrightarrow$ (4). Let $\varepsilon > 0$. Then by (3) $\exists$ partition P_0' s.t

$$U(P_0, f) - L(P_0, f) < \frac{\varepsilon}{3} \quad \ldots \quad (***)$$

Let $n_0 = \#$ of points in P_0, $P_0 = \{s_0, \ldots, s_{n_0}\}$

For a $\delta > 0$ to be <u>chosen later</u>, let P be <u>any</u> partition with $|P| < \delta$.

Also, let $M = \sup_{t \in [a,b]} |f(t)|$, and $P^* = P_0 \cup P$.



<u>Goal</u>: $\exists$ $\delta > 0$ s.t $U(P,f) - L(P,f) < \varepsilon$.

Remember: P was an arbitrary partition.

Now,

$$\begin{aligned} U(P,f) - L(P,f) = {} & U(P,f) - U(P^*,f) \\ & + U(P^*,f) - L(P^*,f) \\ & + L(P^*,f) - L(P,f). \end{aligned}$$

$(***)$ & Lemma 5.2 $\Rightarrow$ (Since $P_0 \subseteq P^*$)

$$U(P^*, f) - L(P^*, f) < \frac{\varepsilon}{3}$$

<u>Claim</u>: $\exists\ \delta > 0$ (ind. of P) s.t

$$U(P, f) - U(P^*, f) < \frac{\varepsilon}{3}$$

$$L(P^*, f) - L(P, f) < \varepsilon/3$$

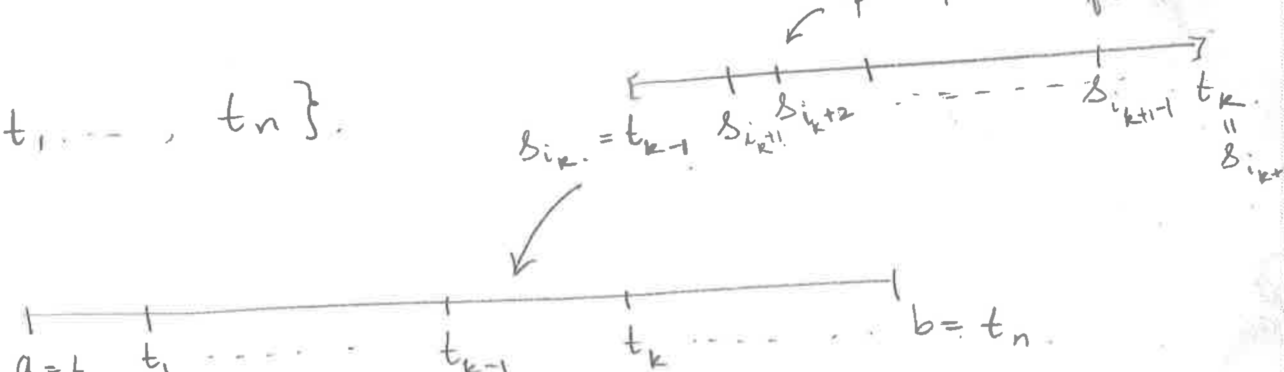
Assuming claim, then $\forall\ P$ s.t $|P| < \delta$,

$$U(P, f) - L(P, f) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

Done!

<u>Pf of Claim</u>: We prove 2^{nd} ineq. Pf of 1^{st} is similar

$P = \{t_0 < t_1, \ldots, t_n\}$.



Let P_k^* be the partition of $[t_{k-1}, t_k]$ by points in P^* (and hence by points in P_0).

Then

$$L(P^*, f) = \sum_{k=1}^{n} L(P_k^*, f)$$

Moreover, $L(P, f) = \sum_{k=1}^{n} m_k \Delta t_k$, $m_k = \inf_{[t_{k-1}, t_k]} f(t)$.

So
$$L(P^*, f) - L(P, f) = \sum_{k=1}^{n} \left[L(P_k^*, f) - m_k \Delta t_k \right]$$

Idea: Classify indices $\{1, 2, \ldots, n\}$ into good indices G & bad indices B i.e.
$\{1, 2, \ldots, n\} = G \cup B$, $G \cap B = \emptyset$ s.t.

$$k \in G \iff P_k^* = \{t_{k-1}, t_k\}$$
$$\iff (t_{k-1}, t_k) \cap P_0 = \emptyset.$$

For $k \in G$, $L(P_k^*, f) - m_k \Delta t_k = 0$ ..... (i)

Also
$\#$ of bad indices $= |B| < n_0$,
where $n_0 = \#$ of elements in P_0.

Moreover, for each k,
$$L(P_k^*, f) \leq M(t_k - t_{k-1}) = M \Delta t_k, \text{ and}$$
so,
$$L(P_k^*, f) - m_k \Delta t_k \leq (M - m_k) \Delta t_k$$
(ii)
$$< 2M \cdot \delta \quad \text{since } |P| < \delta$$
and $M - m_k \leq 2M$.

where recall $M = \sup_{[a,b]} |f(t)|$

So $\forall k$, (and hence all $k \in B$),

$$0 < L(P_k^*, f) - m_k \Delta t_k < 2M \cdot \delta .$$

$$\Rightarrow L(P^*, f) - L(P, f) = \sum_{k=1}^{n} L(P_k^*, f) - m_k \Delta t_k$$

$$= \sum_{k \in B} \left[L(P_k^*, f) - m_k \Delta t_k \right]$$

$$< 2M \cdot \delta \sum_{k \in B} 1$$

$$< 2M \cdot \delta \cdot n_0 .$$

Choose $\delta = \varepsilon / 6 M n_0$.

Then $L(P^*, f) - L(P, f) < \varepsilon / 3$.

and claim is proved.

<u>$(4) \Rightarrow (1)$</u>: Let $r = \mathcal{U}(f)$.

<u>Claim</u>: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$|P| < \delta \Rightarrow | S(f, P, \{t_k^*\}) - r | < \varepsilon .$$

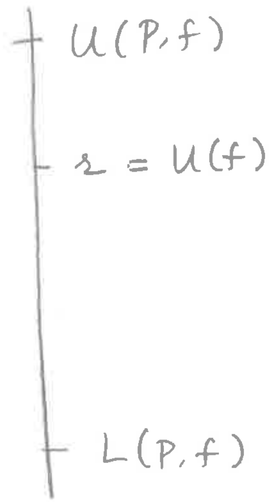
for any choice of points $\{t_k^*\}$.

Assuming the claim, clearly $f \in \mathcal{R}[a, b]$ with

$$\mathcal{R}\int_a^b f(t)\,dt = r .$$

<u>Pf of claim</u>: Let $\varepsilon > 0$. By (4) $\exists$ $\delta > 0$ s.t

$$|P| < \delta \implies U(P,f) - L(P,f) < \varepsilon.$$



Since $U(f) \geq L(f)$, clearly for any P

$$L(P,f) \leq r \leq U(P,f). \quad \text{(iii)}$$

Moreover, if $P = \{t_0, \dots, t_n\}$, then for any $t_k^* \in (t_{k-1}, t_k)$,

$$m_k \leq f(t_k^*) \leq M_k$$

So,

$$L(P,f) \leq S(f, P, \{t_k^*\}) \leq U(P,f) \quad \text{(iv)}$$

If $|P| < \delta$, then $U(P,f) - L(P,f) < \varepsilon$

Then (iii) & (iv) $\implies |r - S(f, P, \{t_k^*\})| < \varepsilon$.

So $f \in R[a,b]$ with $R\int_a^b f(t)\,dt = r$

<u>Rk</u>: From now on integrability means Riemann or Darboux integrability <u>interchangeably</u>. For historical reasons, we continue to write $f \in R[a,b]$, though in most applications, we will use $U(P,f)$ & $L(P,f)$ instead of $S(f, P, \{t_k^*\})$

· <u>Continuity and integrability</u>.

<u>Th^m 5.4</u>: Let $f : [a,b] \to \mathbb{R}$ be bounded & monotonic. Then $f \in R[a,b]$.

<u>Pf</u>: Let $\varepsilon > 0$. For n big, let

$$P = \{t_0 = a, t_1 = a + \frac{b-a}{n}, t_2 = a + \frac{2(b-a)}{n}, \dots, t_n = b\}$$

Sps $f \uparrow$. Then $M_k = \sup_{[t_{k-1}, t_k]} f(t) = f(t_k)$

$$m_k = \inf_{[t_{k-1}, t_k]} f(t) = f(t_{k-1})$$

Also $\Delta t_k = 1/n$. So,

$$U(P,f) - L(P,f) = \sum_{k=1}^{n} (M_k - m_k) \Delta t_k$$

$$= \frac{1}{n} \sum_{k=1}^{n} f(t_k) - f(t_{k-1})$$

$$= \frac{1}{n} \left[f(\cancel{t_1}) - f(t_0) + f(\cancel{t_2}) - f(\cancel{t_1}) + \dots + f(t_n) - f(\cancel{t_{n-1}}) \right]$$

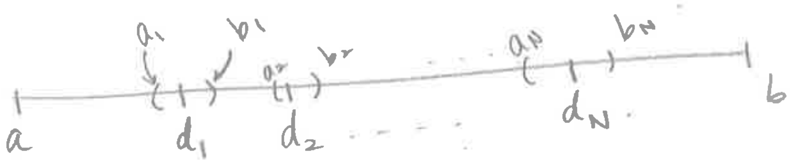
$$= \frac{f(b) - f(a)}{n} < \varepsilon$$

if n is chosen big enough. Then Th^m 5.1 (3) $\Longrightarrow$ $f \in R[a,b]$.

Th^m 5.5: Let $f : [a,b] \to \mathbb{R}$ bounded. If f has only finitely many discontinuities, then $f \in R[a,b]$.

Pf: Let $D \subset [a,b]$ be the set of discont-inuities. Let $D = \{d_1 < d_2 < \dots < d_N\}$. For simplicity sps $d_1 \neq a$, $d_N \neq b$.

Let $\varepsilon > 0$, and $\eta > 0$ to be chosen later to depend on ε. Let $M = \sup|f(t)|$.



Let (a_j, b_j) be disjoint intervals around d_j s.t.

$$\sum_{j=1}^{N} (b_j - a_j) < \eta \qquad (*)$$

Let $B = \bigcup_{j=1}^{N} (a_j, b_j)$ be the "bad set".

and $G = [a,b] \setminus B$, the "good set".

Then $G = \bigcup_{\ell=0}^{N} J_\ell$, where each J_ℓ is a closed interval.

$J_0 = [a, a_1]$, $J_\ell = (b_\ell, a_{\ell+1})$, $\ell = 1, \dots, N-1$.

$J_N = [b_N, b]$.

f cont. on $J_\ell \implies f$ is uniformly cont. on J_ℓ. $\exists\ \delta_\ell > 0$ s.t

$$\left.\begin{array}{l}|s-t| < \delta_\ell \\ s,t \in J_\ell\end{array}\right\} \implies |f(s) - f(t)| < \eta.$$

Let $\delta = \min(\delta_0, \delta_1, \dots, \delta_N)$. Then

$$\left.\begin{array}{l}|s-t| < \delta \\ s,t \in G\end{array}\right. \implies |f(s) - f(t)| < \varepsilon. \quad (**)$$

<u>Goal</u>: Construct partition P s.t

$$U(P,f) - L(P,f) < \varepsilon.$$

Let $P = \{t_0, t_1, \dots, t_n\}$ s.t

(1) $a_j, b_j \in P$.

(2) $(a_j, b_j) \cap P = \emptyset$

(3) If $t_{k-1} \neq a_j$, (hence $t_k \neq b_j$). then

$\Delta t_k < \delta$.

So essentially slice up G into intervals of length $< \delta$. The end points of these 'along' with a_j, b_j make up P.

Note $\forall\ k$,

(a) $0 \leq M_k - m_k \leq 2M$.

(b) $\forall\ k$ s.t $t_{k-1} \in G$ (and hence $(t_{k-1}, t_k) \subset G$)

$$M_k - m_k < \eta.$$

$\Longrightarrow$

$$U(P,f) - L(P,f) = \sum_{k=1}^{n} (M_k - m_k)\, \Delta t_k$$

$$= \sum_{t_{k-1} \neq a_j} (M_k - m_k)\, \Delta t_k + \sum_{t_{k-1} = a} \overbrace{(M_k - m_k)}^{\leq 2M} \Delta t_k$$

$$\leq \eta \sum_{t_{k-1} \neq a_j} \Delta t_k + 2M \sum (b_j - a_j).$$

$$< \eta (b-a) + 2M\eta = \eta (b - a + 2M)$$

Simply choose $\eta = \dfrac{\varepsilon}{b-a+2M}$.

Then $\exists$ P (which we constructed) s.t

$$U(P,f) - L(P,f) < \varepsilon.$$

The theorem can be generalized further, although the proof requires some topological ideas about compactness, that we are yet to cover

Def^n A subset $A \subseteq \mathbb{R}$ is said to be of measure zero, denoted by $\mu(A) = 0$, if $\forall\ \varepsilon > 0$, $\exists$ a sequence of open intervals $\{J_\ell = (a_\ell, b_\ell)\}_{\ell=1}^{\infty}$ (allowing for $J_\ell = \phi$ for some ℓ).

$$(1)\quad A \subseteq \bigcup_{\ell=1}^{\infty} J_\ell$$

$$(2)\quad \sum_{\ell=1}^{\infty} (b_\ell - a_\ell) < \varepsilon$$

Example 1) Let A be finite, say $A = \{P_1, \cdots P_N\}$

Then let $\eta < \varepsilon/2N$ and consider.

$$J_\ell = (P_\ell - \eta,\ P_\ell + \eta),\ \ell = 1, \cdots, N.$$

$$J_\ell = \phi,\quad \ell > N$$

Then $A \subseteq \bigcup J_\ell$, and

$$\sum_{\ell=1}^{\infty} \text{length}(J_\ell) = 2N\eta < \varepsilon.$$

2) Let $S = \{p_1, p_2, \ldots\}$ be any seq^n in $\mathbb{R}$.

Claim $\mu(S) = 0$.

Pf: Given any $\varepsilon > 0$, consider the intervals

$$J_\ell = \left(p_\ell - \frac{\varepsilon}{4^\ell},\ p_\ell + \frac{\varepsilon}{4^\ell}\right).$$

Then length$(J_\ell) = 2\varepsilon/4^\ell$. Clearly $S \subseteq \bigcup_{\ell=1}^{N} J_\ell$.

Also,

$$\sum_{\ell=1}^{\infty} \text{length}(J_\ell) = 2\varepsilon \sum_{\ell=1}^{\infty} \frac{1}{4^\ell}$$

$$= \frac{\varepsilon}{2} \sum_{\ell=0}^{\infty} 4^{-\ell} = \frac{\varepsilon}{2} \cdot \frac{1}{1 - 1/4}$$

$$= \frac{2\varepsilon}{3} < \varepsilon.$$

So $\mu(S) = 0$.

3) $\mu(\mathbb{Q}) = 0$.

FACT: $\mathbb{Q}$ is countable i.e we can write the set of all rationals as a list

$$\mathbb{Q} = \{q_1, q_2, \ldots\}.$$

2) $\Rightarrow$ $\mu(\mathbb{Q}) = 0$.

Th^{m} 5.6 (Lebesgue). Let $f : [a,b] \to \mathbb{R}$ and D_f denote the set of discont. of f. Then
$$f \in R[a,b] \iff \mu(D_f) = 0.$$

Rk: If D_f s.t $\forall \varepsilon > 0$, $D_f \subseteq \bigcup_{l=1}^{N} J_l$ s.t $\sum \operatorname{len}(J_l) < \varepsilon$, then our earlier proof works just fine. But if N cannot be taken finite, for e.g. if $D_f = \mathbb{Q} \cap [a,b]$, then one needs some topological inputs to make the proof go through.

Ex: Let
$$f(x) = \begin{cases} \frac{1}{n}, & x = \frac{m}{n} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

Then for any interval $[a,b]$, $f \in R[a,b]$. since we saw in Assignment-3 that f is discont. only on $\mathbb{Q}$, & by 3), $\mu(\mathbb{Q}) = 0$. So Th^{m} 5.6 $\Rightarrow$ $f \in R[a,b]$.

· Properties of integration

Thm 5.7 1) If $f_1, f_2 \in R[a,b]$ & c_1, c_2 then $c_1 f_1 + c_2 f_2$ $\in R[a,b]$ and

(linearity) $$\int_a^b c_1 f_1 + c_2 f_2 = c_1 \int_a^b f_1 + c_2 \int_a^b f_2.$$

2) If $f_1 \leq f_2$ on $[a,b]$, & $f_1, f_2 \in R[a,b]$, then

$$\int_a^b f_1 \leq \int_a^b f_2.$$

3) $f \in R[a,b]$ and $a < c < b$, then $f \in R[a,c]$ and $f \in R[c,b]$, and

$$\int_a^b f(t)\,dt = \int_a^c f(t)\,dt + \int_c^b f(t)\,dt.$$

Pf: Left as an exercise (see Assignment-5).

Thm 5.8: Sps $f \in R[a,b]$ and $m \leq f \leq M$ on $[a,b]$. Sps $\varphi : [m, M] \longrightarrow \mathbb{R}$ is continuous. Then $\varphi \circ f \in R[a,b]$.

Rk Continuity of φ is crucial. Consider

$$f(x) = \begin{cases} 1/n, & x = m/n \\ 0, & x \notin \mathbb{Q} \end{cases}, \quad \varphi(x) = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0. \end{cases}$$

Then $$\varphi \circ f = \begin{cases} 1, & x \notin \mathbb{Q} \\ 0, & x \in \mathbb{Q}. \end{cases}$$

We have already seen that $\varphi \circ f \notin R[a,b]$ for any interval $[a,b]$.

Rk: The proof of Th^m 5.8, uses ideas similar to ones used in proofs of Th^m 5.1 & Th^m 5.5 (namely good set / bad set decompositions), but is a bit more tricky. Interested readers can refer to my "Integration" notes from Fall 2017.

Cor 5.9: (1) $f, g \in R[a,b] \implies f \cdot g \in R[a,b]$.

(2) $f \in R[a,b] \implies |f| \in R[a,b]$. Moreover

(triangle ineq) $$\left| \int_a^b f(t)\,dt \right| \leq \int_a^b |f(t)|\,dt$$

Pf: (1) Apply Th^m to $\varphi(x) = x^2$. Then f^2, $g^2 \in R[a,b]$. So Th^m 5.7 $\Rightarrow$ $(f+g)^2 \in R[a,b]$

& $$f \cdot g = \frac{1}{2}\left[(f+g)^2 - f^2 - g^2\right] \in R[a,b]$$

(2) Apply Th^m to $\varphi(x) = |x|$. Then $|f| \in R[a,b]$

Let $c = \operatorname{sgn}\left(\int_a^b f(t)\,dt\right)$ i.e. $c = \begin{cases} 1, & \int \geq 0 \\ -1, & \int < 0. \end{cases}$

Then
$$\left|\int_a^b f(t)\,dt\right| = c \cdot \int_a^b f(t)\,dt$$
$$= \int_a^b c \cdot f\,dt$$
$$\leq \int_a^b |f|\,dt$$

since $c = \pm 1$, and $c \cdot f \leq |f|$.

· Integration and Differentiation

Th^m 5.10: (1st fundamental Th^m) If $f \in R[a,b]$. consider $F: [a,b] \longrightarrow \mathbb{R}$ defined by

$$F(x) = \int_a^x f(t)\,dt$$

Then

(a) F is cont. on $[a,b]$.

(b) If f is cont. at p, then F is diff. at p. Moreover

$$F'(p) = f(p).$$

Pf: (a) Sps $|f(t)| \leq M$ $\forall$ $t \in [a,b]$. Let $x, y \in [a,b]$

Claim: $|F(y) - F(x)| \leq M|y-x|$.

Pf: W.l.o.g sps $y > x$. Then

$$|F(y) - F(x)| = \left| \int_a^y f(t)\,dt - \int_a^x f(t)\,dt \right|$$

$$= \left| \int_x^y f(t)\,dt \right| \leq \int_x^y |f(t)|\,dt$$

$$\leq M \int_x^y dt = M(y-x) = M|y-x|$$

Now, given $\varepsilon > 0$, let $\delta = \varepsilon/M$. Then

$$\left.\begin{array}{l} |x-y| < \delta \\ x, y \in [a,b] \end{array}\right\} \Longrightarrow |F(x) - F(y)| \leq M|y-x| < M\delta = \varepsilon.$$

(b) Let $\varphi(x) = [F(x) - F(p)]/(x-p)$ be the diff. quotient of F at p.

If $x > p$, then

$$\varphi(x) = \frac{1}{x-p} \int_p^x f(t)\,dt.$$

Also, since $\int_p^x 1\,dt = x - p$, we have

$$f(p) = \frac{1}{x-p} \int_p^x f(p)\,dt.$$

So $$|\varphi(x) - f(p)| = \left| \frac{1}{x-p} \int_p^x [f(t) - f(p)]\,dt \right.$$

So for all $x > p$,

$$|\varphi(x) - f(p)| \leq \frac{1}{x-p} \int_p^x |f(t) - f(p)|\,dt.$$

Let $\varepsilon > 0$, f cont. at $p \Rightarrow \exists\ \delta > 0$ s.t.

$|t-p| < \delta \Rightarrow |f(t) - f(p)| < \varepsilon$.

Sps $x \in (p, p+\delta)$. Then $\forall\ t \in (p, x)$,

$$|f(t) - f(p)| < \varepsilon.$$

So $$\int_p^x |f(t) - f(p)|\,dt < \varepsilon(x-p).$$

So $\forall\ x \in (p, p+\delta)$, $|\varphi(x) - f(p)| < \varepsilon$.

$\Rightarrow \varphi(p+) = f(p)$.

Similarly, $\varphi(p-) = f(p)$ and so $\lim_{x\to p} \varphi(x)$ exist & equals $f(p)$

$\Rightarrow$ F is diff. at p & $F'(p) = f(p)$

<u>Def^n</u>: A diff. function $F : [a,b] \longrightarrow \mathbb{R}$ is called an <u>anti-derivative</u> or <u>indefinite integral</u> of f, denoted by $F = \int f$ if

$$F'(x) = f(x)$$

$\forall\ x \in (a,b)$.

<u>Rk</u>: If F is an anti-derivative of f on $[a,b]$, then so is $F + c$, for any const. $c \in \mathbb{R}$.

Conversely, if F & G are anti-derivatives of f on $[a,b]$, then $(F-G)' = 0$ on $[a,b]$ and so $G = F + c$, for some const. $c \in \mathbb{R}$.

<u>Ex</u>: If $f(x) = x^n$, $n \neq -1$, then $F(x) = x^{n+1}/n+1$ is an anti-derivative. In fact

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$$

is the set of all possible anti-derivative

$n = -1$, $d\ln x/dx = 1/x$. In fact for $x \neq 0$, $d\ln|x|/dx = 1/x$, so

$$\int \frac{dx}{x} = \ln|x|.$$

Cor 5.11 (2^{nd} fundamental Th^m). If f is cont. on $[a,b]$ & F an anti-derivative.

then

$$\int_a^b f(t)\,dt = F(b) - F(a) \overset{\text{(notation)}}{:=} F(t)\Big|_{t=a}^{b}$$

Pf: Let $G(x) = \int_a^x f(t)\,dt$. Th^m 5.10 $\Longrightarrow$

G is diff. on (a,b), & cont. on $[a,b]$. Moreover
$G(a) = 0$, $G(b) = \int_a^b f(t)\,dt$. & $G'(x) = f(x)$.
Since $F'(x) = f(x)$ $\forall\, x \in (a,b)$,

$$G(x) = F(x) + C, \quad C \in \mathbb{R}$$

But then

$$F(b) - F(a) = G(b) - G(a) = \int_a^b f(t)\,dt$$

Done!

Example: 1) $\int \sin x\, dx = -\cos x + C$, since
$(\cos x)' = -\sin x$. So then Cor 5.11 $\Longrightarrow$

$$\int_a^b \sin x\, dx = -\cos b + \cos a$$

2) For any $\alpha \in \mathbb{R}$

$$\int x^\alpha \, dx = \begin{cases} \dfrac{x^{\alpha+1}}{\alpha+1}, & \alpha \neq -1 \\ \ln|x|, & \alpha = -1. \end{cases}$$

So, for instance.

$$\int_a^b \sqrt{x}\, dx = \left. \frac{x^{3/2}}{3/2} \right|_a^b = \frac{2}{3}\left(b^{3/2} - a^{3/2}\right).$$

• Applications of the fundamental theorems

Th^m 5.12 (Integration by parts). If u, v are cont. functions on $[a, b]$, & if u' and v' are cont. on $[a, b]$. Then

$$\int_a^b u(x)\, v'(x)\, dx = u(b)\,v(b) - u(a)\,v(a) - \int_a^b v(x)\, u'(x)\, dx.$$

Rk 1) Easier to remember using language of differentials. Let $dv := v'(x)\,dx$ and $du = u'(x)\,dx$

Then in indefinite form:

$$\int u\, dv = uv - \int v\, du + C.$$

2) As you will see in the proof, you should

think of this as the inverse of the product rule of differentiation

Pf of Th^m: Let $F(x) = uv$; then $F' = u \cdot v' + v \cdot u'$.

Since F' is cont. on $[a,b]$, Cor 5.11

$$\Longrightarrow \int_a^b F'(x)\,dx = F(b) - F(a) = u(b)v(b) - u(a)v(a)$$

But $$\int_a^b F'(x)\,dx = \int_a^b u(x)\,v'(x)\,dx + \int_a^b v(x)\,u'(x)\,dx$$

Done!

Example 1) $\int \ln t\,dt$, $u = \ln t$, $du = dt/t$, $dv = dt$, $v = t$.

So $$\int \ln t\,dt = t\ln t - \int dt + C$$
$$= (t-1)\ln t + C$$

2) $\int_0^{\pi} \underbrace{x}_{u}\,\underbrace{\cos x\,dx}_{dv}$, $u = x$, $du = dx$, $dV = \cos x\,dx$, $V = \sin x$.

So $$\int_0^{\pi} x\cos x\,dx = x\sin x\Big|_0^{\pi} - \int_0^{\pi}\sin x\,dx$$
$$= \cos x\Big|_0^{\pi} = -2$$

$\underline{\text{Th}^m\ 5.13}$ (Change of variables / u-sub). Let $I \subseteq \mathbb{R}$ open interval & $g : I \to \mathbb{R}$ s.t g' <u>exist</u> and is <u>cont</u> on I. Let $J = g(I)$ & $f : J \to \mathbb{R}$ cont. Then $\forall\ a, b \in I$,

$$\int_a^b f(g(t)) \cdot g'(t)\, dt = \int_{g(a)}^{g(b)} f(u)\, du.$$

<u>Rk</u>: In terms of differentials, one should think of this as $u = g(t)$, $du = g'(t)\, dt$.

<u>Pf</u>: Let

$$G(t) = \int_{g(a)}^{g(t)} f(u)\, du, \quad F(u) = \int_{g(a)}^{u} f(s)\, ds.$$

Then $G(t) = F(g(t))$. Since f is cont. 1^{st} fundamental $\text{Th}^m \Rightarrow F$ is diff. & $F'(u) = f(u)$. Also g is diff. So, G is diff and by chain rule

$$G'(t) = F'(g(t)) \cdot g'(t)$$
$$= f(g(t)) \cdot g'(t).$$

2^{nd} fund. $\text{Th}^m \Rightarrow \int_a^b f(g(t)) \cdot g'(t) = G(b) - G(a).$

Clearly $G(a) = 0$ and $G(b) = \int_{g(a)}^{g(b)} f(u)\,du$, and hence.

$$\int_a^b f(g(t))\cdot g'(t)\,dt = \int_{g(a)}^{g(b)} f(u)\,du.$$

Rk: 1) As an indefinite integral, we can set

$$\int f(g(t))\cdot g'(t)\,dt = \int f(u)\,du.$$

2) u-sub is the inverse of chain rule.

Ex: $\displaystyle\int_0^{\pi/4} \tan t\,dt = \int_0^{\pi/4} \frac{\sin t}{\cos t}\,dt$, $\quad u = \cos t = g(t)$. $\quad du = -\sin t\,dt$.

$$= -\int_1^{1/\sqrt{2}} \frac{du}{u} = \int_{1/\sqrt{2}}^{1} \frac{du}{u} = \ln u \Big|_{1/\sqrt{2}}^{1}$$

$$= \frac{\ln 2}{2}.$$

• Improper integrals. Till now we have restricted attention to

(1) $[a, b]$ bounded interval

(2) f bounded.

Def^n: Let $-\infty \le a < b \le \infty$. We say f is locally integrable on (a,b), denoted by $f \in R_{loc}(a,b)$ if $\forall$ $a < c < d < b$, $f \in R[c,d]$. We say f is integrable on (a,b) if it is locally integrable, and

$$\lim_{c \to a^+} \left[\lim_{d \to b^-} \int_c^d f(t)\,dt \right] := \int_a^b f(t)\,dt.$$

exists and is finite.

Def^n: We say f has an improper integral on (a,b) or that $\int_a^b f$ converges if f is integrable on (a,b) & one of the foll. holds

(1) $a = -\infty$.

(2) $b = \infty$.

(3) f is unbounded on (a,b).

We denote this by $f \in IR(a,b)$.

FACT: If f is integrable on (a,b), then

$$\lim_{c \to a^+} \left[\lim_{d \to b^-} \int_c^d f(t)\,dt \right] = \lim_{d \to b^-} \left[\lim_{c \to a^+} \int_c^d f(t)\,dt \right]$$

Else, we say $\int_a^b f$ diverges.

Ex: 1) $\int_a^\infty x^{-p}$, $a > 0$.

Clearly $x^{-p} \in R[a,b]$ $\forall$ $b < \infty$. Moreover

$$\int_a^b x^{-p}\,dx = \begin{cases} \dfrac{a^{-p+1}}{p-1} - \dfrac{b^{-p+1}}{p-1}, & p \neq 1 \\ \ln b - \ln a, & p = 1 \end{cases}$$

If $p = 1$, $\lim_{b\to\infty} \int_a^b x^{-1}\,dx = \infty$, so $\int_a^\infty x^{-1}\,dx$ diverges

If $p \neq 1$, $\lim_{b\to\infty} \int_a^b x^{-p}\,dx$ exists and is $\dfrac{a^{1-p}}{p-1}$ if and only if $-p+1 < 0$ or $p > 1$.

So

$$\int_a^\infty x^{-p}\,dx = \begin{cases} \dfrac{a^{1-p}}{p-1}, & p > 1 \\ \text{diverges}, & p \leq 1. \end{cases}$$

2) $\int_0^a x^{-p}\,dx$, $a > 0$.

Clearly $x^{-p} \in R[c, a]$ $\forall$ $c \in (0, a)$.

By the same reasoning as above.

$$\int_0^a x^{-p}\,dx = \begin{cases} \dfrac{a^{1-p}}{1-p}, & p < 1. \\ \text{diverges}, & p \geq 1. \end{cases}$$

e.g. $p = 1/2$.

$$\int_0^a \frac{dx}{\sqrt{x}} = \lim_{c \to 0^+} \int_c^a \frac{dx}{\sqrt{x}} = \lim_{c \to 0^+} 2\sqrt{x}\Big|_c^a$$

$$= 2\sqrt{a} - \lim_{c \to 0^+} 2\sqrt{c} = 2\sqrt{a}.$$

Many of the theorems for convergent series also hold for convergent integrals.

Thm 5.14: Let $f, g : (a, b) \longrightarrow \mathbb{R}$, $-\infty \leq a < b \leq \infty$. Then

(1) (Cauchy criteria) (i) If $a \neq -\infty$, $b \neq \infty$. Then

$\int_a^b f$ converges $\Longleftrightarrow$ $\forall \varepsilon > 0$, $\exists\ \delta$ s.t $\forall\ c_1, c_2 \in (a, a+\delta)$, $\forall\ d_1, d_2 \in (b-\delta, b)$,

$$\int_{c_1}^{c_2} f + \int_{d_1}^{d_2} f < \varepsilon.$$

(ii) If $b = \infty$, $a \neq -\infty$ then $\int_a^\infty f$ conv $\Longleftrightarrow$ $\forall \varepsilon > 0$,

$\exists\ M$ s.t $\forall\ c, d > M$,

$$\int_c^d f < \varepsilon.$$

(iii) Similarly for $a = -\infty$.

(2) (Absolute conv). $\int_a^b |f|$ conv $\Longrightarrow \int_a^b f$ conv.

(3) (Sps. $f \geq 0$. Then

$$\sup_{[a,b)} |f| < \infty \Longrightarrow \int_a^b f \text{ conv.}$$

(4) ((Comparison). Sps $0 \leq f \leq g$ $\forall x \in (a,c) \cup (d,b)$. for some $a < c < d < b$

(i) $\int_a^b g$ conv. $\Longrightarrow \int_a^b f$ conv.

(2) $\int_a^b f$ div $\Longrightarrow \int_a^b g$ div.

(5) (Limit Comparison). Sps. $f, g \geq 0$ and $f, g \in R[a, c]$ $\forall$ $c < b$. Sps

$$\lim_{x \to b^+} \frac{f(x)}{g(x)} = L \neq 0, \infty$$

Then $\int_a^b f$ conv. $\Longleftrightarrow \int_a^b g$ conv.

One can formulate a similar statement about $f, g \in R[c, b]$ $\forall$ $a < c$.

<u>Ex</u>: (1) (Gamma function).

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1}\, dx.$$

Claim: Integral on the right converges for all $s > 0$

Pf: Note

$$\int_0^\infty e^{-x} x^{s-1}\, dx := \int_0^1 e^{-x} x^{s-1} dx + \int_1^\infty e^{-x} x^{s-1} dx$$

For $x \in [0,1]$, $e^{-1} x^{s-1} \leq x^{s-1} e^{-x} \leq x^{s-1}$

Since $\int_0^1 x^{s-1} dx$ conv $\Leftrightarrow s-1 > -1$.

$\Leftrightarrow s > 0$.

Comparison test $\Longrightarrow \int_0^1 e^{-x} x^{s-1} dx$ conv. $\Leftrightarrow s > 0$.

Now, for $x \geqslant 1$,

Claim: $\forall s > 0$ $\exists M_s$ s.t. $\forall x > M_s$.

$$0 \leq e^{-x} x^{s-1} \leq e^{-x/2}.$$

Pf: For any real $a>0$, p, $\lim\limits_{x\to\infty} \frac{x^p}{a^x} = 0$

Now $\int_1^\infty e^{-x/2}$ conv. (Check this). So.

Comparison $\Longrightarrow \int_1^\infty e^{-x} x^{s-1} dx$ conv.

This proves the claim

(2) $\displaystyle\int_0^\infty \left|\frac{\sin x}{x}\right| dx$ diverges. but $\displaystyle\int_0^\infty \frac{\sin x}{x}$ converges

(see homework).