WEEK - 4

• Basic properties of differentiable functions

Th^m 4.2 If $f:(a,b)\to\mathbb{R}$ is differentiable at a point $p\in(a,b)$ Then f is cont. at p.

Pf. We are given that $f'(p) = \lim\limits_{x\to p} \dfrac{f(x)-f(p)}{x-p}$

exists. Now, for $x \neq p$

$$f(x) = (x-p)\,\frac{f(x)-f(p)}{x-p} + f(p).$$

By product rule for limits

$$\lim_{x\to p} f(x) = \lim_{x\to p}(x-p)\cdot \lim_{x\to p}\frac{f(x)-f(p)}{x-p} + f(p)$$

$$= 0\cdot f'(p) + f(p) = f(p)$$

$\Rightarrow$ f is cont at p

Th^m 4.3 (Alg. rules) Let $f, g: (a,b)\to\mathbb{R}$ and $p\in(a,b)$. If f, g are diff at p, then each of the functio[ns] cf ($c\in\mathbb{R}$), $f+g$ & $f\cdot g$ are also diff at p. If $g(p)\neq 0$, then f/g is also diff at p Moreover

(1) $(cf)'(p) = c\cdot f'(p)$

(2) $(f+g)'(p) = f'(p) + g'(p)$.

(3) [product rule] $(f\cdot g)'(p) = f(p)\, g'(p) + g(p)\, f'(p)$.

(4) [quotient rule] $\left(\frac{f}{g}\right)'(p) = \dfrac{g(p)\, f'(p) - f(p)\, g'(p)}{g^2(p)}$

Pf: We prove (3) & (4). The other parts are simpler.

(3) Note that for $x \neq p$

$$\frac{f\cdot g(x) - f\cdot g(p)}{x-p} = f(x)\,\frac{g(x)-g(p)}{x-p} + g(p)\,\frac{f(x)-f(p)}{x-p}$$

$$\xrightarrow{x \to p} f(p)\cdot g'(p) + g(p)\cdot f'(p).$$

where we used Th^m 4.2 to conclude $\lim_{x\to p} f(x) = f(p)$

(4) Since $g(p) \neq 0$ and g is cont! $\exists$ an interval $I = (p-\delta, p+\delta)$ s.t $\forall\ x \in I$, $g(x) \neq 0$.

Then, for $x \in I$.

$$(f/g)(x) - (f/g)\,p = \frac{g(p)\, f(x) - g(x)\, f(p)}{g(x)\, g(p)}$$

$$= \frac{g(p)\,[f(x) - f(p)] - f(p)\,[g(x) - g(p)]}{g(x)\, g(p)}$$

So for $x \in I$, $x \neq p$,

$$\frac{(f/g)(x) - (f/g)(p)}{x-p} = \frac{g(p)\left[\frac{f(x)-f(p)}{x-p}\right] - f(p)\left[\frac{g(x)-g(p)}{x-p}\right]}{g(x)\,g(p)}$$

$$\xrightarrow{x \to p} \frac{g(p)\,f'(p) - f(p)\,g'(p)}{g(p)^2}.$$

Thm 4.3 (Chain rule) Let f be diff. at p, and g be diff at $f(p)$, then $g \circ f$ is diff at p, and

$$g \circ f'(p) = g'(f(p)) \cdot f'(p).$$

"Incorrect proof": Write

$$\frac{g \circ f(x) - g \circ f(p)}{x-p} = \frac{g(f(x)) - g(f(p))}{f(x)-f(p)} \cdot \frac{f(x)-f(p)}{x-p}$$

As $x \to p$, $f(x) \to f(p)$, and so

$$\lim_{x \to p} \frac{g(f(x)) - g(f(p))}{f(x)-f(p)} = g'(f(p)).$$

and so

$$\frac{g(f(x)) - g(f(p))}{f(x) - f(p)} \cdot \frac{f(x) - f(p)}{x - p} \longrightarrow g'(f(p)) \cdot f'(p).$$

__Problem__: There might be a seqⁿ $x_n \to p$ s.t $f(x_n) = f(p)$. That is one might not be able to divide by $f(x) - f(p)$.

__e.g__: $g(x) = x$, $f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0. \end{cases}$

Then. $f\left(\frac{1}{n\pi}\right) = 0 \quad \forall n$. So with $p = 0$

$$\frac{g\circ f(x) - g\circ f(0)}{f(x) - f(0)}$$

is undefined for values of x arbitrarily close to $p = 0$.

__Pf of Thᵐ 4.3__ Let $E = \{x \mid f(x) = f(p)\}$, and define

$$\psi(x) = \begin{cases} \dfrac{g\circ f(x) - g\circ f(p)}{f(x) - f(p)}, & x \in E^c. \\ g'(f(p)), & x \in E. \end{cases}$$

__Claim__ $\psi(x)$ is cont

Assuming claim we note that $\lim_{x\to p} \psi(x) = \psi(p) = g'(f(p))$

$$g\circ f(x) - g\circ f(p) = \psi(x)[f(x) - f(p)]$$

$\forall\, x$, not just ones in E^c. So for $x \neq p$,

$$\frac{g\circ f(x) - g\circ f(p)}{x-p} = \psi(x)\left[\frac{f(x) - f(p)}{x-p}\right]$$

$$\xrightarrow{x\to p} \psi(p) f'(p)$$

$$= g'(f(p))\cdot f'(p)$$

Pf of Claim: Let $x_n \longrightarrow x_0$.

Goal: Prove that $\lim_{n\to\infty} \psi(x_n) = \psi(x_0)$.

CASE I: $x_0 \in E^c$. Then $f(x_0) \neq f(p)$. Since f is cont, $\exists\ \delta$ s.t. $\forall\ x \in (x_0 - \delta, x_0 + \delta)$, $f(x) \neq f(p)$.

To see this, apply def^n of cont. to $\varepsilon = \frac{|f(x_0) - f(p)|}{2}$

Then $\exists\ \delta$ s.t. $|x - x_0| < \delta \Longrightarrow$

$$|f(x) - f(x_0)| < \varepsilon.$$

If $x \in (x_0 - \delta, x_0 + \delta)$ & $f(x) = f(p)$, then

$$|f(x_0) - f(p)| = |f(x_0) - f(x)| < \varepsilon = \frac{|f(x_0) - f(p)|}{2}$$

Contradiction. In particular $\exists\ N$ s.t. $\forall\ n > N$,

$f(x_n) \neq f(p)$ or $\Longleftrightarrow$ $x_n \in E^c$.

So for $n > N$.

$$\psi(x_n) = \frac{g\circ f(x_n) - g\circ f(p)}{f(x_n) - f(p)} \underset{=}{\overset{y_n = f(x_n)}{}} \frac{g(y_n) - g(f(p))}{y_n - f(p)}$$

$$\xrightarrow[y_n \to f(x_0)]{n\to\infty} \frac{g(f(x_0)) - g(f(p))}{f(x_0) - f(p)} = \psi(x_0).$$

<u>CASE 2</u> $x_0 \in E$. Let $\{x_{n_k}\}$ subsequence s.t $n_k \notin E$. Since $\psi(x_0) = g'(f(p))$, enough to show that $\lim_{k\to\infty} \psi(x_{n_k}) = g'(f(p))$.

Since for any x_n s.t $n \neq n_k$, $x_n \in E$ and
So $\psi(x_n) = g'(f(p))$.

Now, since $x_{n_k} \in E^c$,

$$\psi(x_{n_k}) = \frac{g(f(x_{n_k})) - g(f(p))}{f(x_{n_k}) - f(p)} \underset{=}{\overset{y_k = f(x_{n_k})}{}} \frac{g(y_k) - g(f(p))}{y_k - f(p)}.$$

f cont $\Rightarrow \lim_{k\to\infty} y_k = f(x_0) = f(p)$ (since $x_0 \in E$).

So

$$\lim_{k\to\infty} \psi(x_{n_k}) = \lim_{k\to\infty} \frac{g(y_k) - g(f(p))}{y_k - f(p)} = g'(f(p)).$$

Done!

Examples 1) Consider $f_1(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0. \end{cases}$

We have seen earlier that $f_1(x)$ is cont. on $\mathbb{R}$.

Ques: Is it diff. on $\mathbb{R}$?

For $x \neq 0$, by product & chain rule, f_1 is diff and,

$$f_1'(x) = \sin\left(\frac{1}{x}\right) + x \cdot \cos\left(\frac{1}{x}\right) \cdot \left(\frac{-1}{x^2}\right)$$

$$= \sin(1/x) - x^{-1} \cos(1/x).$$

At $x = 0$: The diff. quotient is

$$\varphi(t) = \frac{f(t) - f(0)}{t} = \frac{t \sin(1/t)}{t} = \sin(1/t).$$

So $\lim_{t \to 0} \varphi(t)$ DNE & f is NOT diff. at $x = 0$.

2) Consider $f_2(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0. \end{cases}$

Again, when $x \neq 0$, f_2 is diff by product & chain rule, and

$$f_2'(x) = 2x \sin\left(\frac{1}{x}\right) + x^2 \cos\left(\frac{1}{x}\right) \cdot \left(\frac{-1}{x^2}\right)$$

$$= 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right).$$

<u>Note</u>: $\lim_{x \to 0} f_2'(x)$ DNE. But

<u>Claim</u>: $f_2'(0)$ exists

<u>Pf</u>: The diff quotient is

$$\varphi(t) = \frac{f(t) - f(0)}{t} = t \sin\left(\frac{1}{t}\right) \xrightarrow{t \to 0} 0$$

So $f_2'(0) = 0$

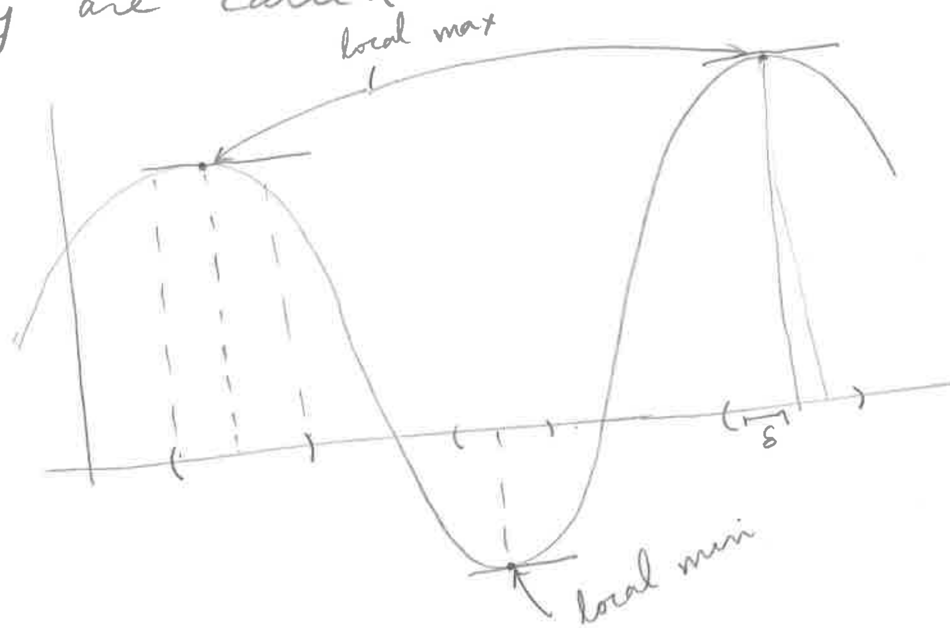
<u>Rk</u>:
Since $\lim_{x \to 0} f_2'(x)$ DNE, $f_2'(x)$ has an <u>essential</u> <u>discont.</u> at $x = 0$.

- Local extremas

Def^n Let $f: I \to \mathbb{R}$, where I is an interval. We say $p \in I$ is a local max (resp local min) if $\exists$ $\delta > 0$ s.t. $\forall$ $x \in (p-\delta, p+\delta) \cap I$,

$$f(x) \leq f(p) \quad (\text{resp } f(x) \geq f(p))$$

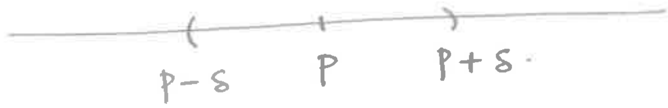
Together they are called local extremas.



Th^m 4.4 Let $I \subseteq \mathbb{R}$ be an interval and $f: I \to \mathbb{R}$ have a local extrema at an interior point $p \in I$. If f is diff at $p \implies f'(p) = 0$

Pf: Sps p is a local max (Similar argument also works for local min). Then $\exists$ $\delta > 0$ s.t.

$$f(p) \geq f(t) \quad \forall \quad t \in (p-\delta, p+\delta)$$



If $t \in (p, p+\delta)$, $\varphi(t) := \dfrac{f(t)-f(p)}{t-p} \leq 0$

$t \in (p-\delta, p)$, $\varphi(t) = \dfrac{f(t)-f(p)}{t-p} \geq 0$

So $\varphi(p+) \leq 0$, $\varphi(p-) \geq 0$

f diff at p $\Rightarrow$ $f'(p) = \varphi(p+) = \varphi(p-)$

So $f'(p) = 0$

Rk: By the extremum value theorem any cont. $f: [a,b] \to \mathbb{R}$ has a global max 'p' and min 'q'

Thm 4.4 $\Rightarrow$ One of the foll holds

(1) $p = a$ or b

(2) f is not diff at p $\Big\}$ p is called a critical point

(3) $f'(p) = 0$

Similarly for q.

Ex Consider $f(t) = (1-t^2)^2$ on $[-3, 3]$.

Then f is diff on $[-2, 2]$.

<u>Critical points</u> $f'(t) = 2(1-t^2)(-2t)$

$$= -4t(1-t^2)$$

So critical points are $t=0$, $t=\pm 1$.

<u>Function values</u>: $f(0) = 1$, $f(\pm 1) = 0$.

End points: $f(\pm 3) = 64$

So Max is at ± 3 and min at ± 1.

• <u>Mean Value theorems</u>

<u>Th^m 4.5</u> (Rolle's Theorem) Let $f : [a, b] \longrightarrow \mathbb{R}$ be continuous on $[a, b]$ and diff. on (a, b). If $f(a) = f(b)$, then $\exists\ c \in (a, b)$ s.t $f'(c) = 0$.

<u>Pf</u>: EVT $\Longrightarrow$ $\exists\ p, q \in [a, b]$ s.t

$$f(p) \leq f(x) \leq f(q)$$

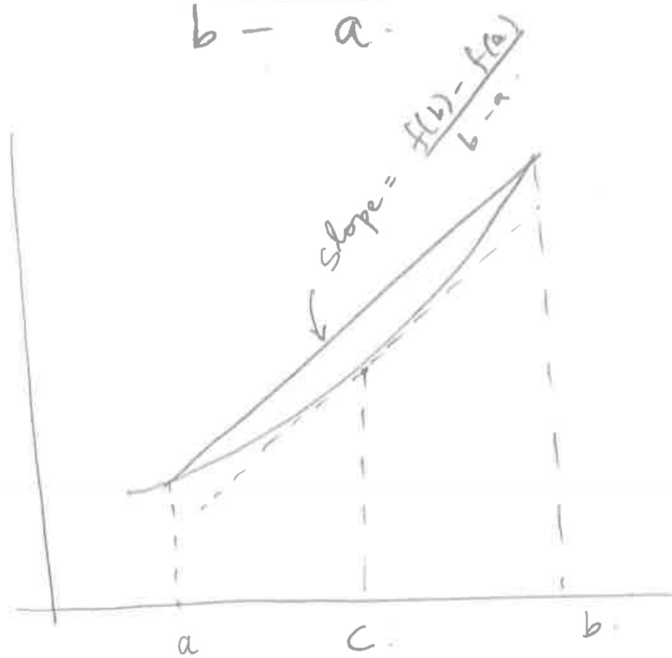
$\forall\ x \in [a, b]$. <u>If</u> both p and q are end points of $[a, b]$, since $f(a) = f(b)$, f has to be a constant. Then $f'(x) = 0\ \forall\ x$.

<u>Else</u> at least one, say $p \in (a, b)$. Then

Th^m 4.4 $\Longrightarrow$ $f'(p) = 0$.

<u>Th^m 4.6</u> (Mean value theorem, MVT) Let f be cont on $[a, b]$ and diff on (a, b). Then $\exists\ c \in (a, b)$ s.t

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



<u>Pf</u> Consider

$$g(t) = [f(b) - f(a)]\,t - (b-a)\,f(t).$$

Then (1) g is cont. on $[a, b]$.

(2) g is diff on (a, b).

Rolle's $\Longrightarrow$ $\exists\ c \in (a, b)$ s.t $g'(c) = 0$

But $g'(t) = [f(b) - f(a)] - (b-a)\,f'(t)$

So $g'(c) = 0 \Longrightarrow f'(c) = \dfrac{f(b) - f(a)}{b - a}$

<u>Rk</u> f has to be cont. on all of $[a,b]$.

Else, consider

$$f(x) = \begin{cases} x, & 0 < x \le 1 \\ 1, & x = 0 \end{cases}$$

Then $f'(x) = 1$ on (a,b) while $f(1) - f(0) = 0$.

<u>Cor 4.7</u>. Let f be cont. on $[a,b]$ and diff. on (a,b). Then

(1) f is constant $\iff f'(x) = 0 \quad \forall\, x \in (a,b)$

(2) $f \uparrow \iff f'(x) \geq 0 \quad \forall\, x \in (a,b)$.

(3) $f \downarrow \iff f'(x) \leq 0 \quad \forall\, x \in (a,b)$.

<u>Pf</u>: (1) $\Longrightarrow$ trivial.

$\Longleftarrow$ If not, then $\exists\ p, q \in [a,b]$ s.t. $f(p) \neq f(q)$. Then. MVT $\Longrightarrow \exists\ c$ s.t.

$$0 \overset{\text{hypothesis}}{=} f'(c) = \frac{f(p) - f(q)}{p - q} \neq 0$$

Contradiction!

(2) $\Longrightarrow$ Sps $f \uparrow$. Consider diff. quotient

$$\varphi(t) = \frac{f(t) - f(x)}{t - x}.$$

If $t > x$, $f(t) \geq f(x)$

$t < x$, $f(t) \leq f(x)$

In both cases, $\varphi(t) \geq 0$.

Then $f'(x) = \lim_{t \to x} \varphi(t) \geq 0$.

$\Leftarrow$ If not, then $\exists\ p, q \in [a,b]$ s.t $p < q$ but $f(p) > f(q)$. MVT $\Longrightarrow$ $\exists\ c \in (p,q)$ s.t

$$0 \overset{\text{hypothesis}}{\leq} f'(c) = \frac{f(q) - f(p)}{q - p} < 0.$$

Contradiction.

(3) Same as (2) above.

• Cor 4.8 (IVT for diff.). Let $f : (a,b) \longrightarrow \mathbb{R}$ diff. If $a < x_1 < x_2 < b$ & c lies between $f'(x_1)$ & $f'(x_2)$ Then $\exists\ x_0 \in (x_1, x_2)$ s.t

$$f'(x_0) = c.$$

Pf: Let $g(x) = f(x) - cx$.

Then $g'(x) = f'(x) - c$, and $g'(x_1) < 0$, $g'(x_2) > 0$

g is cont. on $[x_1, x_2]$, so $\exists\ x_0 \in [x_1, x_2]$ s.t $g(x_0)$ is a min.

Claim $x_0 \neq x_1, x_2$

Pf Since $g'(x_1) < 0$ &

$$g_1'(x_1) = \lim_{t \to x_1} \frac{g(t) - g(x_1)}{t - x_1},$$

$\exists$ $\delta > 0$ s.t $\forall$ $t \in (x_1, x_1 + \delta)$, $g(t) < g(x_1)$

So x_1 cannot be min. for g on $[x_1, x_2]$

i.e $x_1 \neq x_0$. Similarly $x_0 \neq x_2$. Hence $x_0 \in (x_1, x_2)$

and $g'(x_0) = 0$. Then $f'(x_0) = c$.

- <u>Differentiability of f^{-1}</u>

Sps $f: I \longrightarrow \mathbb{R}$ be one-one, cont and diff.

s.t $f^{-1}: f(I) \longrightarrow I$ is also diff. Then.

by definition

$$f^{-1} \circ f(x) = x.$$

Differentiating both sides. w.r.t 'x' at $x = x_0$,

by <u>chain rule</u>, if $y_0 = f(x_0)$.

$$(f^{-1})'(y_0) \cdot f'(x_0) = 1.$$

In particular, $f'(x_0) \neq 0$ &

$$\boxed{(f^{-1})'(y_0) = \frac{1}{f'(x_0)}}$$

Conversly, we have the foll theorem (without proof)

Th^m 4.9: Let f: be one-one cont. on an open interval $I \subseteq \mathbb{R}$. Let $J = f(I)$. If f is diff at $x_0 \in I$ & $f'(x_0) \neq 0$, then f^{-1} is diff at $y_0 = f(x_0) \in J$ and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$

Ex: 1) $f(x) = x^3$ on $\mathbb{R}$. Then f is 1-1 and cont.

$$f^{-1}(y) = y^{1/3}.$$

Claim: f^{-1} is NOT diff. at $y = 0$.

Pf: Difference quotient

$$\varphi(t) = \frac{f^{-1}(t) - f^{-1}(0)}{t} = \frac{t^{1/3}}{t} = t^{-2/3} \xrightarrow{t \to 0} \text{DNE}$$

Problem is that $f'(0) = 0$, so Th^m 4.9 does not apply.

2) (Logarithms) Let $f(x) = e^x$ on $\mathbb{R}$. Then

$e^x = f'(x) > 0$ so f is <u>strictly</u> $\uparrow$

In particular, f is one-one.

The inverse function is $\ln y : (0, \infty) \longrightarrow \mathbb{R}$

Thm 4.9 $\Rightarrow$ $\ln x$ is diff everywhere, and

at $y = e^x$, $$\frac{d}{dy} \ln y = \frac{1}{de^x/dx} = \frac{1}{e^x} = \frac{1}{y}.$$

$$\boxed{(\ln y)' = \frac{1}{y}}$$

<u>Application</u>: Let $f(x) = x^\alpha$, $x > 0$, $\alpha \in \mathbb{R}$

Then $\ln f(x) = \alpha \cdot \ln x$.

Diff, by chain rule

$$\frac{f'(x)}{f(x)} = \frac{\alpha}{x}$$

So $$f'(x) = \alpha \cdot \frac{f(x)}{x} = \alpha x^{\alpha - 1}$$

So $$\boxed{\frac{d}{dx} x^\alpha = \alpha x^{\alpha - 1}}$$

L'Hospital's rule:

Thm 4.10 (Generalized MVT) Let f, g be cont on $[a, b]$ & diff on (a, b). Then $\exists$ $c \in (a, b)$ s.t

$$f'(c)[g(b) - g(a)] = g'(c)[f(b) - f(a)].$$

Rk Recover MVT, by letting $g(x) = x$.

Pf of Thm: Let

$$h(x) = f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)]$$

Then (1) h is cont on $[a, b]$ & diff on (a, b).

(2) $h(a) = f(a)g(b) - f(b)g(a)$
$= h(b)$.

Rolle's Thm $\Longrightarrow$ $\exists$ $c \in (a, b)$ s.t $h'(c) = 0$.

Done!

Thm 4.11 (L'Hospital's rule). Let s signify p, p^+, p^-, ∞ or $-\infty$; where $p \in \mathbb{R}$. Sps f and g are diff functions for which

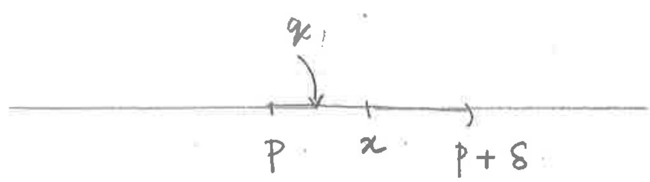
$$\lim_{x \to s} \frac{f'(x)}{g'(x)} = L \qquad (*)$$

If (i) $\lim_{x \to s} f(x) = \lim_{x \to s} g(x) = 0$

Or (ii) $\lim_{x \to s} |g(x)| = \lim_{x \to s} |f(x)| = \infty$.

Then
$$\lim_{x \to s} \frac{f(x)}{g(x)} = L.$$

Pf We give a proof in case of (i) & when $s = p^{+} \in \mathbb{R}$. Other cases are similar.



Let $\varepsilon > 0$ $(*) \Longrightarrow \exists \ \delta > 0$ s.t

$$\left| \frac{f'(t)}{g'(t)} - L \right| < \varepsilon \quad \forall \ t \in (p, p+\delta). \quad (**)$$

Let $x \in (p, p+\delta)$ and q s.t $p < q < x$.

Then generalized MVT $\Longrightarrow \exists \ t \in (q, x)$ s.t

$$\frac{f'(t)}{g'(t)} = \frac{f(x) - f(q)}{g(x) - g(q)}$$

** $\Longrightarrow$

$$\left| \frac{f(x) - f(q)}{g(x) - g(q)} - L \right| < \varepsilon$$

Let $q \to p$. Then $\lim_{q \to p} f(q) = \lim_{q \to p} g(q) = 0$.

So $$\left| \frac{f(x)}{g(x)} - L \right| < \varepsilon.$$

Given $\varepsilon > 0$, $\exists\ \delta > 0$ s.t

$$x \in (p, p+\delta) \Longrightarrow \left| \frac{f(x)}{g(x)} - L \right| < \varepsilon.$$

So $$\lim_{x \to p^+} \frac{f(x)}{g(x)} = L.$$

Examples. 1) $\displaystyle \lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{(\sin x)'}{(x)'} = \lim_{x \to 0} \cos x = 1$

2) $\displaystyle \lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{\sin x}{2x} = \frac{1}{2}$.

3) (Pitfall). Consider $\displaystyle \lim_{x \to 0} \frac{\ln x}{x}$.

Blindly applying L'Hospital, $\displaystyle \lim_{x \to 0^+} \frac{\ln x}{x} = \lim_{x \to 0^+} \frac{1/x}{1}$

$= \infty$

WRONG!

Since as $x \to 0$, $\ln x \to -\infty$, $\ln x / x \to -\infty/0$.

& L'Hospital cannot directly apply.

Instead,

$$\frac{\ln x}{x} = -\frac{\ln 1/x}{x} = -y \ln y, \quad \text{if } y = 1/x.$$

As $x \to 0^+$, $y \to +\infty$, $\ln y \to \infty$. So

$$\frac{\ln x}{x} \xrightarrow{x \to \infty} -\infty$$

• Taylor's Theorem: If f is diff at $p \in (a, b)$, then

$$r(h) = \frac{f(p+h) - f(p)}{h} - f'(p) \xrightarrow{h \to 0} 0$$

In other words, if $x \approx p$, then

$$f(x) \approx f(p) + f'(p)(x-p).$$

Def^n: We define the linearization of f at p by

$$L(x) = f(p) + f'(p)(x-p)$$

Then if $x \approx p$, $f(x) \approx L(x)$.

Taylor's theorem is a generalization of this.

We denote the <u>higher</u> order derivatives of f by $f', f'', \dots, f^{(n)}$ etc if they exist

i.e if $f', f'', \dots, f^{(k-1)}$ exist, then

$$f^{(k)}(p) = \lim_{x \to p} \frac{f^{(k-1)}(x) - f^{(k-1)}(p)}{x - p}$$

if the limit exist.

<u>Note</u>: If $f', \dots, f^{(n)}$ exist at p, then $f', \dots, f^{(n-1)}$ are cont.

<u>Def^n</u> Let $f', f'', \dots, f^{(n)}$ exist at some $p \in (a, b)$

The n^{th} Taylor polynomial at $x = p$ is defined as

$$T_n(p; x) = T_n(x) := \sum_{k=0}^{n} \frac{f^{(k)}(p)}{k!} (x - p)^k,$$

and the n^{th} remainder is defined as

$$R_n(p; x) = R_n(x) = f(x) - T_n(x).$$ (Notation diff from Ross).

<u>Rk</u> 1) $L(x) = T_1(x)$

2) If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$

Then if $p = 0$, $f^{(k)}(0) = k! \cdot a_k$, so

$$f(x) = T_n(0; x).$$

<u>Th^m</u> 4.12 Let $f:(a,b) \longrightarrow \mathbb{R}$ and $p \in (a,b)$. (We allow $a = -\infty$ or $b = \infty$). Sps $f^{(n)}$ exists on (a,b). Then $\forall\ x \in (a,b)$, $x \neq p$, $\exists$ a c between x and p.

s.t

$$f(x) = T_{n-1}(p;x) + \frac{f^{(n)}(c)}{n!}(x-p)^n.$$

Consequently, if $M = \sup\limits_{(a,b)} |f^{(n)}(t)| < \infty$, then

(Taylor's est.) $$|R_{n-1}(p;x)| \leq \frac{M}{n!}|x-p|^n$$

<u>Rk</u>: If $n=1$, this is the MVT.

<u>Pf</u>: Fix $x \in (a,b)$, $x \neq p$, and let

$$Q = \frac{f(x) - T_{n-1}(p;x)}{(x-p)^n}.$$

<u>Goal</u>: $\exists$ c between p & x s.t $Q = f^{(n)}(c)/n!$

Consider
$$g(t) = f(t) - T_{n-1}(p;t) - Q(t-p)^n.$$

<u>Claim</u> $\exists$ c bet. p and x s.t $g^{(n)}(c) = 0$.

Note that $$\frac{d^n}{dt^n} T_{n-1}(p;t) = 0$$

since T_{n-1} is a deg $(n-1)$ polynomial

So, $\quad g^{(n)}(t) = f^{(n)}(t) - Q \cdot n!$

Claim $\Rightarrow \exists\ c$ s.t $\quad f^{(n)}(c) - Qn! = 0$ or $Q = \dfrac{f^{(n)}(c)}{n!}$

So, enough to prove claim.

Pf of Claim: Note that g has the foll. properties.

(1) $g^{(k)}(p) = 0, \quad k = 0, 1, \ldots, n-1$

(2) $g(x) = 0$

Rolle's Th^m $\Rightarrow \exists\ x_1$ between p and x s.t $g'(x_1) = 0$. Since $g'(p) = 0$, again by Rolle's $\exists\ x_2$ bet. x_1 and p s.t $g''(x_2) = 0$ and so on. Process continues until we have x_n between p and x_{n-1} s.t $g^{(n)}(x_n) = 0$. Since all $x_1, \ldots, x_{n-1}$ lie between p and x by construction, x_n also lies between p and x, so Claim is proved for $c = x_n$.

For the est. note that

$$R_{n-1}(x) = T_{n-1}(p; x) = \frac{f^{(n)}(c)}{n!}(x-p)^n;$$

where $c \in (a, b)$. If $M = \sup\limits_{t \in (a,b)} |f^{(n)}(t)|$, then

$$|R_{n-1}(x)| = \frac{|f^{(n)}(c)|}{n!}|x-p|^n \leq \frac{M|x-p|^n}{n!}$$

- Applications to optimization

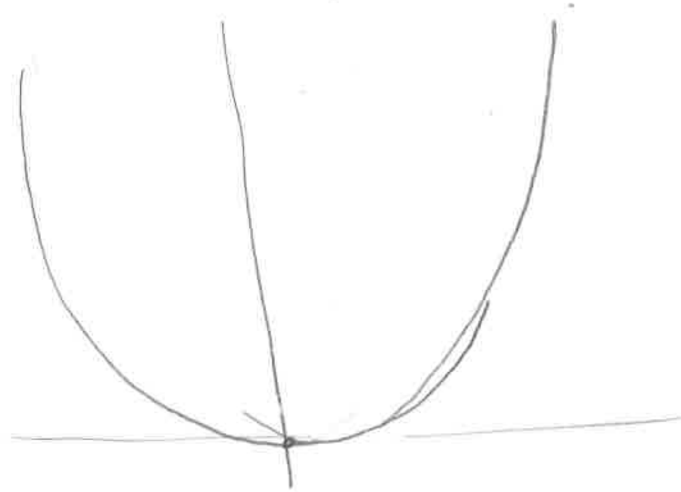
Def^n. Let $f:(a,b) \to \mathbb{R}$. We say $f \in C^n[a,b]$ if $f', \ldots, f^n$ exist on (a,b) and are uniformly cont. on (a,b).

Recall: If $p \in (a,b)$ is a local max / min, then $f'(p) = 0$.

Ques: How do you decide if it is local max / min

In calculus we study 2^{nd} derivative test. But that has limitations.

Consider $f(x) = x^4$ on $\mathbb{R}$



Clearly f has a local min at $x=0$, but $f^{(2)}(0) = 0$.

Th^m 4.13 Let $f \in C^n[a,b]$ and $p \in (a,b)$ s.t

$f'(p) = f''(p) \cdots = f^{(n-1)}(p) = 0$, but $f^{(n)}(p) \neq 0$.

Then.

(1) n is even, $f^{(n)}(p) > 0 \implies p$ is a local min

(2) n is even, $f^{(n)}(p) < 0 \implies p$ is a local max

(3) n is odd $\implies f$ has neither a local max nor a local min. p is then called an inflection point

Pf (1) Sps n is even and $f^{(n)}(p) > 0$.

Cont. of $f^{(n)} \implies \exists\ \delta > 0$ s.t.

$t \in (p-\delta, p+\delta) \implies f^{(n)}(t) > 0$ (*).

Let $x \in (p-\delta, p+\delta)$. Since $f^{(k)}(p) = 0$ for all $k = 1, 2, \ldots, n-1$, clearly

$$T_{n-1}(p; x) = f(p).$$

So Taylor's theorem $\implies \exists\ c$ between p and x s.t

$$f(x) = f(p) + \frac{f^{(n)}(c)}{n!}(x-p)^n.$$

$c \in (p-\delta, p+\delta) \implies f^{(n)}(c) > 0$.

n even $\implies (x-p)^n > 0$

So, $\forall\ x \in (p-\delta,\ p+\delta)$, $f(x) > f(p)$ and p is a local min.

(2) Similar.

(3) Again as above $\exists$ c between p and x s.t

$$f(x) = f(p) + \frac{f^{(n)}(c)}{n!}(x-p)^n$$

If $f^{(n)}(p) > 0$, then can choose δ s.t $f^{(n)}(c) > 0$

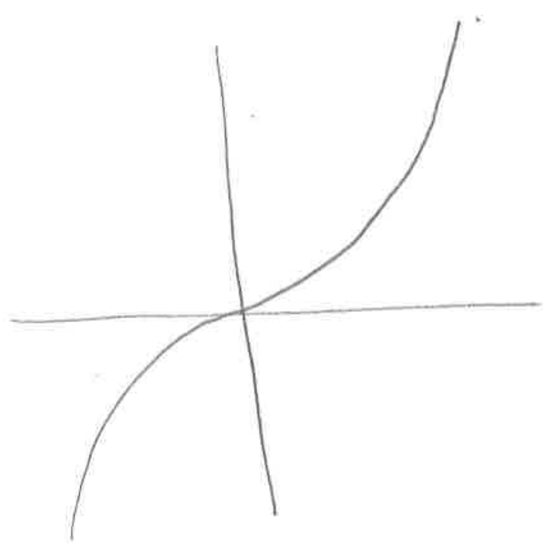
But then n odd, so

$$p < x \Rightarrow f(p) < f(x)$$
$$p > x \Rightarrow f(p) > f(x)$$

and p is neither local max / min

Similar if $f^{(n)}(p) < 0$.

<u>Ex</u> $f(x) = x^3$



$f'(x) = 3x^2$

$f''(x) = 6x$

$f^{(3)}(x) = 6$

So $f'(0) = f^{(2)}(0) = 0$

but $f^{(3)}(0) \neq 0$

3 is odd, so $Th^m \Rightarrow 0$ inflection point

This is clear from graph.

2) Let $f(t) = (1-t^2)^2$. We have seen that the critical points are $p = \pm 1, 0$

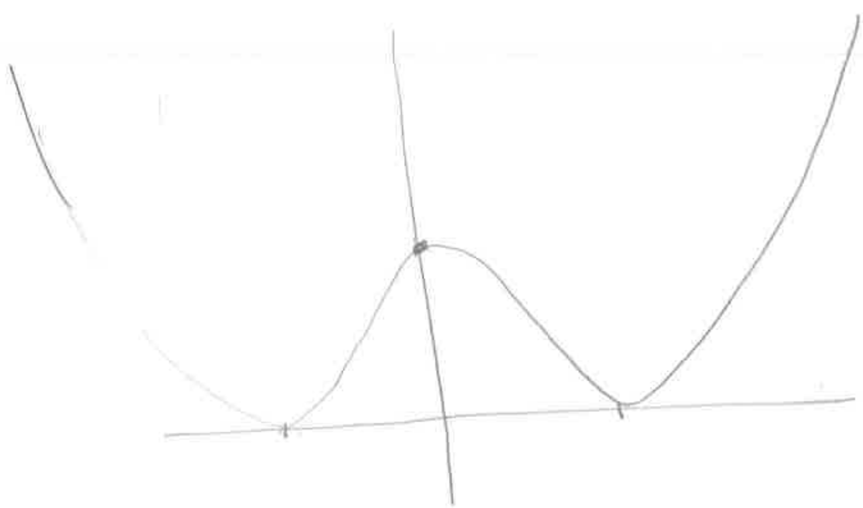
$$f'(t) = -4t(1-t^2)$$

$$f''(t) = -4(1-t^2) + 8t^2.$$

$$= 12t^2 - 4.$$

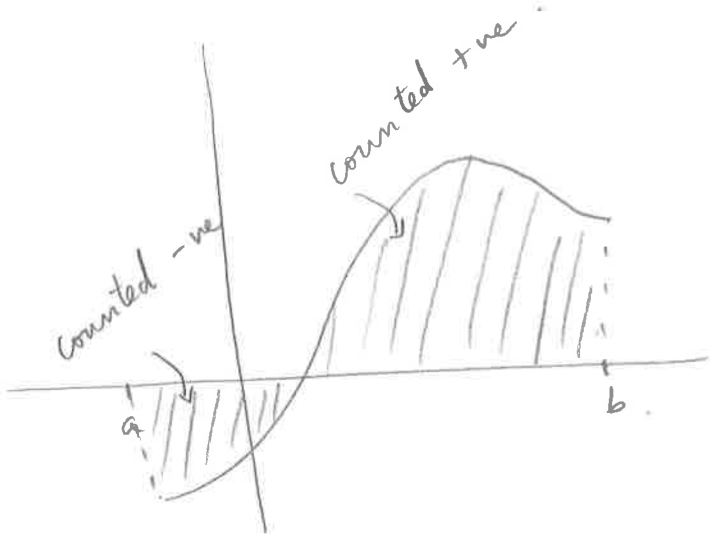
$$f''(\pm 1) = 12 > 0, \quad f''(0) = -4 < 0.$$

So, $p = \pm 1$ are local <u>mins</u> & $p = 0$ is a local max

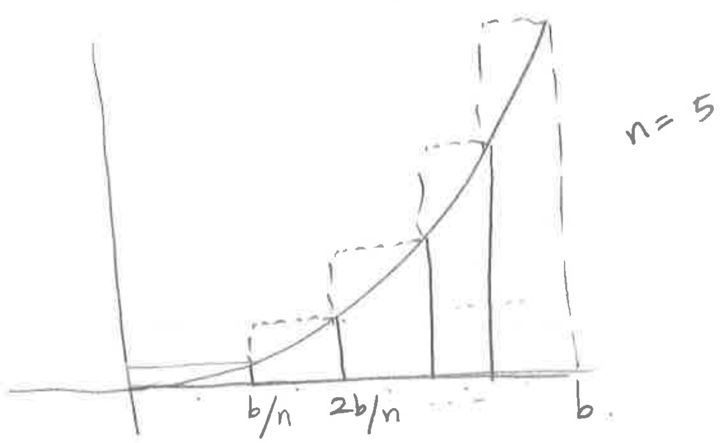


Chapter - 5 Integration

- **Goal**: Given $f: [a,b] \to \mathbb{R}$, compute the (signed) area under $y = f(x)$.



Examples 1) $f(x) = x^2$ on $[0, b]$.



Naive idea: Let $n \in \mathbb{N}$, and partition $[0, b]$ as

$$0 = t_0 < t_1 = \frac{b}{n} < t_2 = \frac{2b}{n} \cdots < t_{n-1} = \frac{(n-1)b}{n} < t_n = b.$$

Then $\Delta t_k = \Delta t = \dfrac{b}{n}$

Let $R_{k,n}$ be the rectangle $[t_{k-1}, t_k] \times [0, f(t_k)]$.

and $$A_{k,n} = \text{Area}(R_{k,n}) = \Delta t_k \cdot f(t_k)$$
$$= t_k^2 \cdot \frac{b}{n}$$
$$= \left(\frac{k \cdot b}{n}\right)^2 \cdot \frac{b}{n}$$
$$= b^3 \cdot \frac{k^2}{n^3}.$$

Total area under rectangles $:= A_n = \sum_{k=1}^{n} A_{k,n}$.

So

$$A_n = \frac{b^3}{n^3} \sum_{k=1}^{n} k^2$$

FACT (Exercise in Assignment - 0).

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

So $$A_n = \frac{b^3}{6} \cdot \frac{(n+1)(2n+1)}{n^2}.$$

Intuitively, if

Area under $y = x^2 := A$, then

$$A = \lim_{n\to\infty} A_n = \frac{b^3}{6} \lim_{n\to\infty} \frac{(n+1)(2n+1)}{n^2}$$

$$= \frac{b^3}{6} \lim_{n\to\infty} \left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

$$= \boxed{\frac{b^3}{3}} \qquad \text{We write } \int_0^b t^2\,dt = \frac{b^3}{3}$$

2) Now, consider

$$f(t) = \begin{cases} 1, & t \in \mathbb{Q} \\ 0, & t \notin \mathbb{Q} \end{cases}$$

If we try to compute $\int_0^1 f(t)\,dt$ using the same method as in (1), then consider

$$0 = t_0 < t_1 = \frac{1}{n} < t_2 = \frac{2}{n} < \cdots < t_n = 1$$

Then $f(t_k) = 1 \quad \forall k$, and $\Delta t_k = \Delta t = 1/n$.

So $$A_n = \sum_{k=1}^{n} f(t_k)\cdot \Delta t_k = \sum_{k=1}^{n} \frac{1}{n} = 1.$$

So the total "area" by this naive idea is 1.

For $\int_0^{\sqrt{2}} f(t)\,dt$, by a similar method, we have $\quad 0 = t_0 < t_1 = \frac{\sqrt{2}}{n} \cdots < t_n = \sqrt{2}$.

Since now, $t_k \notin \mathbb{Q} \ \forall k$, $\quad f(t_k) = 0 \ \forall k$.

So
$$\text{“}\int_0^{\sqrt{2}} f(t)\text{''} \text{“}=\text{''} \ 0 < \text{“}\int_0^1 f(t)\,dt\text{''}$$

But $f(t) \geqslant 0$. So "area" under $y = f(t)$, from 0 to 1 should be $<$ than area from 0 to $\sqrt{2}$.

So there is a problem with this naive idea.

• <u>Riemann integrability</u>. For the rest of the chapter, unless otherwise stated, $f : [a,b] \to \mathbb{R}$ is assumed to be <u>bounded</u>.

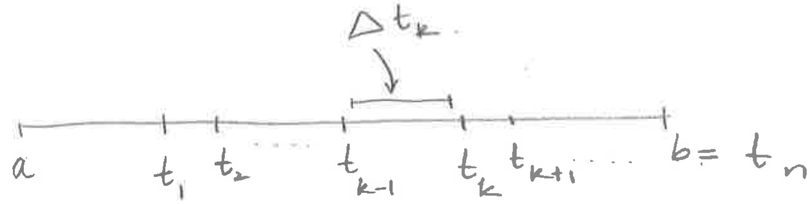
<u>Def^n</u> A <u>partition</u> P of $[a,b]$ is a finite set $\{t_0, t_1, \ldots, t_n\}$ s.t.

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b.$$

We denote $\quad I_k = [t_{k-1}, t_k]$

$$\Delta t_k = t_k - t_{k-1}.$$

$$\text{mesh}(P) = |P| := \max\{\Delta t_k,\ k = 1, \ldots, n\}$$



<u>Def^n</u> Let P be a partition as above, and $f : [a,b] \to \mathbb{R}$. A <u>Riemann sum</u> (R.S) of f associated to P is a sum of the form

$$\sum_{k=1}^{n} f(t_k^*) \Delta t_k \;:=\; S(f, P, \{t_k^*\}).$$

for some $t_k^* \in [t_{k-1}, t_k]$.

<u>Rk</u>: The choice of t_k^* is <u>arbitrary</u>, so there are infinite number of Riemann sums associated to any P.

<u>Def^n</u>: We say that f is <u>Riemann integrable</u> on $[a,b]$, denoted by $f \in \mathcal{R}[a,b]$, if $\exists\, r \in \mathbb{R}$ s.t $\forall\ \varepsilon > 0$, $\exists\ \delta$ with the property that

$$\|P\| < \delta \implies |S(f, P, \{t_k^*\}) - r| < \varepsilon.$$

We then say that r is the Riemann integral of f on $[a,b]$ and write

$$\mathcal{R}\int_a^b f\, dt = r.$$

Ex. Let $f(t) = \begin{cases} 1, & t \in \mathbb{Q} \\ 0, & t \notin \mathbb{Q} \end{cases}$

Given any $n \in \mathbb{N}$, $\exists$ partition P_n s.t $|P_n| = \frac{1}{n}$

namely: $0 = t_0 < t_1 = \frac{1}{n} \cdots < t_n = 1$. i.e $t_k = \frac{k}{n}$

s.t. if $t_k^* = k/n$, then

$$S(f, P_n, \{t_k^*\}) = 1.$$

while if $s_k^* \in \left[\frac{k-1}{n}, \frac{k}{n}\right]$ any irrational, then

$$S(f, P_n, \{s_k^*\}) = 0$$

So $f \notin \mathcal{R}[0,1]$.

Rk: Definition of R-integrability is easy to understand, but difficult to use to prove theorems.

So instead, we introduce Darboux integrability.

Hard exercise. Show $x^2 \in \mathcal{R}[0,b]$ for any $b > 0$. In fact $x^2 \in \mathcal{R}[a,b]$, for any $a < b$.

- Darboux integrability

For a partition $P = \{t_0, \dots, t_n\}$ of $[a, b]$, we denote

$$M_k = \sup_{t \in [t_{k-1}, t_k]} f(t), \quad m_k = \inf_{t \in [t_{k-1}, t_k]} f(t),$$

and define the upper and lower sums by

$$U(P, f) = \sum_{k=1}^{n} M_k \, \Delta t_k .$$

$$L(P, f) = \sum_{k=1}^{n} m_k \, \Delta t_k .$$

If $M = \sup_{[a,b]} f(t)$, $m = \inf_{t \in [a,b]} f(t)$, then $M_k \leq M$ and $m \leq m_k$ $\forall k$, and since $\sum_{k=1}^{n} \Delta t_k = b - a$,

$$m(b-a) \leq L(P, f) \leq U(P, f) \leq M(b-a)$$

for any P.

Def^n: We define the upper and lower Darboux integrals by

$$U(f) = \inf_{P} U(P, f)$$

$$L(f) = \sup_{P} L(P, f).$$

We say that f is <u>Darboux - integrable</u>, & denote it by $f \in D[a,b]$, if $L(f) = U(f)$.

We then define the Darboux integral of f on $[a,b]$ by

$$\int_a^b f(t)\,dt = U(f) = L(f).$$

<u>Rk</u>: Intuitively, $L(f) \leq U(f)$, but this is <u>far from obvious</u> (Why?). We'll give a proof of this next class.

<u>Examples</u> 1). $f(t) = \begin{cases} 1, & t \in \mathbb{Q} \\ 0, & t \notin \mathbb{Q} \end{cases}$

It is clear that for any P of $[0,1]$,

$$U(P,f) = 1, \quad L(P,f) = 0.$$

So,

$$L(f) = 0 \neq 1 = U(f)$$

and $f \notin D[0,1]$.

2) $f(x) = x^2$ on $[0, b]$. Let P_n be the partition $0 = t_0 < t_1 = \frac{b}{n} < \cdots < t_n = b$, as before, since $f \uparrow$.

$$M_k = f(t_k) = \frac{k^2 b^2}{n^2}, \qquad \Delta t_k = \frac{b}{n}$$

$$m_k = f(t_{k-1}) = \frac{(k-1)^2 b^2}{n^2}$$

So $$U(P_n, f) = \sum_{k=1}^{n} \frac{k^2 b^2}{n^2} \cdot \frac{b}{n} = \frac{b^3}{n^3} \sum_{k=1}^{n} k^2$$

$$= \frac{b^3}{6} \frac{(n+1)(2n+1)}{n^2} \searrow \frac{b^3}{3}.$$

So $$U(f) \leq b^3/3.$$

Also, $$L(P_n, f) = \sum_{k=1}^{n} (k-1)^2 \frac{b^3}{n^3}$$

$$= \frac{b^3}{n^3} \frac{(n-1)(n)(2(n-1)+1)}{6}$$

$$= \frac{b^3}{6} \frac{(n-1)(2n-1)}{n^2} \nearrow \frac{b^3}{3}$$

So $$L(f) \geq b^3/3.$$

Since $L(f) \leq U(f)$ (a fact that we'll prove later) $\Longrightarrow$ $L(f) = U(f) = b^3/3$.

So $f \in D[0,b]$ & $\displaystyle\int_0^b f(t)\,dt = \frac{b^3}{3}$.