WEEK - 3

• Uniform Continuity

Recall: $f: E \to \mathbb{R}$ is cont. at $p \in E$ if $\forall\ \varepsilon > 0$.

$\exists\ \delta = \delta(\varepsilon, p) > 0$ s.t

$$\left.\begin{array}{l} |x - p| < \delta \\ x \in E \end{array}\right\} \Longrightarrow |f(x) - f(p)| < \varepsilon.$$

Examples: 1) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = ax + b$, $a \neq 0$. Given $\varepsilon > 0$ if $\delta = \varepsilon/|a|$ then $\forall\ x, p \in \mathbb{R}$

$$|x - p| < \delta \Rightarrow |f(x) - f(p)| < \varepsilon.$$

Note: δ does not depend on p.

2) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. Again sps. $\varepsilon > 0$.

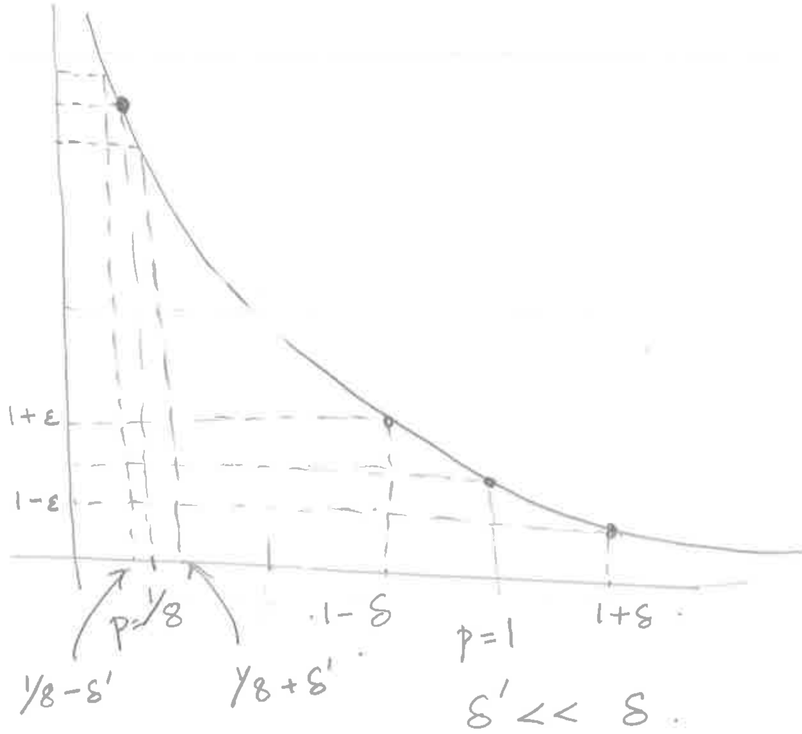
To show cont. at $p \in \mathbb{R}$, we can take any $\delta < \min\left(1, \frac{\varepsilon}{1 + 2|p|}\right)$.

Here δ depends on p. As $|p| \to \infty$, $\delta \to 0$.

3) $f: (0, \infty) \to \mathbb{R}$, $f(x) = 1/x$.

One can show f is cont. on $(0,1)$ using quotient rule.

How does δ depend on ε?



Clearly as $p \to 0$, $\delta \longrightarrow 0$ for same given $\varepsilon > 0$.

Def^n: $f : E \longrightarrow \mathbb{R}$ is said to be <u>uniformly continuous</u> if $\forall\ \varepsilon > 0$, $\exists\ \delta = \delta(\varepsilon)$ s.t. $\forall\ x, y \in E$,

$$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

i.e. δ is independent of y in the definition of uniform cont.

Ex: $f(x) = ax + b$ is uniformly cont. on $\mathbb{R}$.

Rk: If $f : E \to \mathbb{R}$ is <u>not</u> uniformly cont.

then $\exists\ \varepsilon > 0$ s.t $\forall\ \delta > 0$, $\exists\ x_\delta, y_\delta \in E$

s.t

$$|x_\delta - y_\delta| < \delta \quad \underline{\text{but}} \quad |f(x_\delta) - f(y_\delta)| \geq \varepsilon .$$

__Th^m 3.5__ f is not uniformly cont. on $E \iff \exists\ \varepsilon > 0$ s.t $\forall\ n \in \mathbb{N}$, $\exists\ x_n, y_n \in E$ s.t

$$|x_n - y_n| < \frac{1}{n} \quad \text{but} \quad |f(x_n) - f(y_n)| \geq \varepsilon .$$

__Pf.__ $\Longrightarrow$ Apply Rk to $\delta = 1/n$, $n = 1, 2, \ldots$

$\Longleftarrow$ Sps $\exists\ \varepsilon > 0$ s.t $\forall\ n \in \mathbb{N}$, $\exists\ x_n, y_n \in E$ s.t $|x_n - y_n| < \frac{1}{n}$ but $|f(x_n) - f(y_n)| \geq \varepsilon$.

& f is __uniformly cont__. Then $\exists\ \delta$ s.t $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$. Let $n \in \mathbb{N}$ s.t $1/n < \delta$, and x_n, y_n as above. Then $|x_n - y_n| < 1/n < \delta$,

but $|f(x_n) - f(y_n)| \geq \varepsilon$. Contradiction!

__Examples__ (1) $f(x) = x^2$ is NOT uniformly cont. on $\mathbb{R}$.

__Pf.__ Let $x_n = n$, $y_n = n + \frac{1}{2n}$. Then $|x_n - y_n| < \frac{1}{n}$

Also

$$|f(x_n) - f(y_n)| = \left(n + \frac{1}{2n}\right)^2 - n^2 = \frac{1}{4} + \frac{1}{4n^2}$$

Take $\varepsilon = 1/4$.

Then, for $\varepsilon = 1/4$, $\forall\, n \in \mathbb{N}$,

$|x_n - y_n| < \frac{1}{n}$, but $|f(x_n) - f(y_n)| \geq \varepsilon$.

Th^m 3.50 $\Longrightarrow$ f is NOT uniformly cont.

2) $f(x) = 1/x$ is <u>not</u> uniformly cont. on $(0, \infty)$.

<u>Pf</u>: Let $x_n = \frac{1}{n}$, $y_n = \frac{1}{n} + \frac{1}{2n}$. Then $|x_n - y_n|$

$< 1/n$. But

$|f(x_n) - f(y_n)| = n - \frac{2n}{3} = n$.

If $\varepsilon = 1$. Then $\forall\, n$,

$|x_n - y_n| < \frac{1}{n}$ but $|f(x_n) - f(y_n)| \geq \varepsilon$.

Th^m $\Longrightarrow$ f is NOT uniformly cont.

3) $f(x) = 1/x$ on $[1, \infty)$.

<u>Claim</u>: f is uniformly cont. on $[1, \infty)$.

<u>Pf</u>: Let $\varepsilon > 0$.

<u>Goal</u>: Find δ s.t

$|x - y| < \delta \Longrightarrow \left| \frac{1}{x} - \frac{1}{y} \right| < \varepsilon$

Note $\left| \frac{1}{x} - \frac{1}{y} \right| = \frac{|x-y|}{|xy|}$.

If $x, y \in [1, \infty)$, then $|x| \geq 1$, $|y| \geq 1$

So

$$\left| \frac{1}{x} - \frac{1}{y} \right| \leq |x-y|$$

Let $\delta = \varepsilon$. Then

$$|x-y| < \delta \implies \left| \frac{1}{x} - \frac{1}{y} \right| < \varepsilon.$$

In general f is uniformly cont. on $[a, \infty)$ for any $a > 0$. Given $\varepsilon > 0$, we can take $\delta = a^2 \varepsilon$.

<u>Th^m 3.6</u> Let $f : [a,b] \longrightarrow \mathbb{R}$ be cont. Then f is uniformly cont.

<u>Pf</u>: If not. Then $\exists\ \varepsilon > 0$ and seq^{n}s $\{x_n\}$ and $\{y_n\}$ in $[a,b]$ s.t

$$|x_n - y_n| < \frac{1}{n}, \quad \text{but} \quad |f(x_n) - f(y_n)| \geq \varepsilon$$

Bolzano - Weierstrass $\implies \exists\ x_0 \in \mathbb{R}$ & sub-seq $\{x_{n_k}\}$

$x_{n_k} \longrightarrow x_0$,

Since $a \leq x_{n_k} \leq b \ \forall k \implies x_0 \in [a,b]$.

Also, since $|x_{n_k} - y_{n_k}| < \frac{1}{n_k} \xrightarrow{k\to\infty} 0$

$\Rightarrow \lim_{k\to\infty} y_{n_k} = x_0$.

But then f cont. at $x_0 \Rightarrow \lim_{k\to\infty} f(x_{n_k}) = f(x_0)$

$= \lim_{k\to\infty} f(y_{n_k})$.

Contradicting that $|f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon$.

Rk: The theorem is not true if one of the end points a or b is NOT included in the domain. e.g $f(x) = 1/x$ is cont. on $(0,1]$. but not uniformly cont.

· Extremum value property

Def^n: A function $f: E \longrightarrow \mathbb{R}$ is said to be bounded if $f(E)$ is a bounded set in $\mathbb{R}$ i.e $\exists$ M s.t $\forall x \in E$,

$$|f(x)| \leq M.$$

Th^m 3.7 (Extremum value Theorem, EVT). Let $f: [a,b] \to \mathbb{R}$ be cont. Then

(1) f is bounded.

(2) $f(E)$ has a max and min. i.e $\exists$ $x_0, y_0 \in [a,b]$

s.t $f(x_0) \leq f(x) \leq f(y_0)$ $\forall$ $x \in [a,b]$. or equivalently $f(x_0) = \min f(E)$, $f(y_0) = \max f(E)$.

Defⁿ 1) We say $x_0 \in E$ is a (global) min of $f : E \to \mathbb{R}$ if

$$f(x_0) \leq f(x) \quad \forall\, x \in E.$$

2) We say $y_0 \in E$ is a (global) max of $f : E \to \mathbb{R}$ if

$$f(y_0) \geq f(x) \quad \forall\, x \in E.$$

Pf of Thᵐ 1) Sps f is not bounded on $[a,b]$. Then $\forall\, n \in \mathbb{N}$, $\exists$ $x_n \in [a,b]$ s.t

$$|f(x_n)| > n.$$

Bolzano-Weierstrass $\overset{(BW)}{\Longrightarrow}$ $\exists$ sub-seqⁿ x_{n_k} s.t $x_{n_k} \longrightarrow x_0 \in [a,b]$.

$|f|$ cont. $\Longrightarrow$ $|f(x_0)| = \lim\limits_{n \to \infty} |f(x_{n_k})| > \lim\limits_{k \to \infty} n_k = \infty$

Contradiction.

2) Let $M = \sup\{f(x) \mid x \in [a,b]\}$. $M < \infty$ by 1).

Then $\forall$ n $\exists$ $x_n \in [a,b]$ s.t

$$M - \frac{1}{n} < f(x_n) \leq M$$

Again BW $\Longrightarrow$ $\exists$ sub-seqⁿ $x_{n_k} \longrightarrow x_0 \in [a,b]$

Taking $k \to \infty$ in

$$M - \frac{1}{n_k} < f(x_{n_k}) < M$$

since $n_k \to \infty$ and $f(x_{n_k}) \longrightarrow f(x_0)$, squeeze $Th^m \Rightarrow f(x_0) = M$.

So the max is attained. Similarly, we can show that min is attained.

<u>Rk</u> Th^m 2.38 is <u>false</u> if $[a, b]$ is replaced by an open / half open interval. For instance consider $f(x) = 1/x^2$ on $(0, 1]$. As $x \to 0$ f grows unbounded.

<u>Example</u>. Let $f: \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$f(x) = \frac{1}{1+x^2}.$$

Since $x^2 \geq 0$, clearly $0 \leq f(x) \leq 1 \;\; \forall x \in \mathbb{R}$.

Also $f(0) = 1$, so 0 is a max on $\mathbb{R}$.

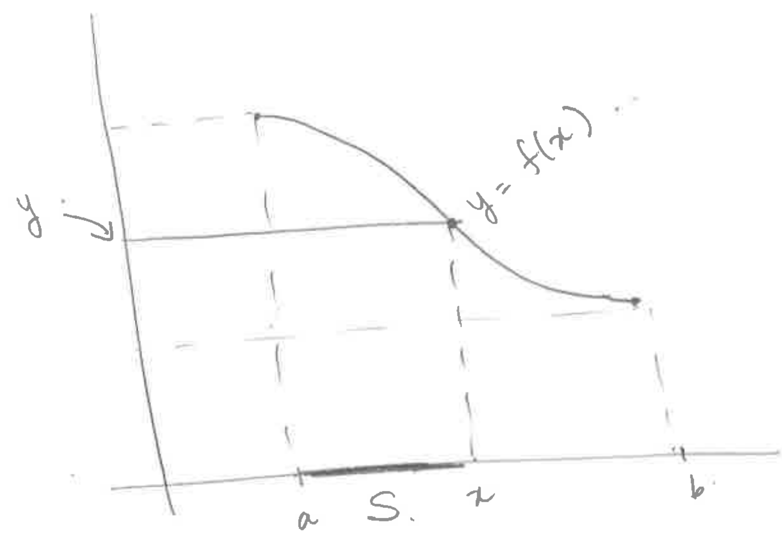
Since $\lim_{x \to \infty} f(x) = 0$, clearly $\inf f(\mathbb{R}) = 0$.

But there is no $x_0 \in \mathbb{R}$ s.t $f(x_0) = 0$. So f has <u>no</u> min on $\mathbb{R}$.

On the other hand, $Th^m \Rightarrow$ on any set $[a,b]$, the function will have both a max & a min.

· Intermediate Value Theorem.

Th^m 3.8 (IVT). If $f: I \to \mathbb{R}$ cont.; and I is an interval, then f has the intermediate value property (IVP) on I i.e. For any $a, b \in I$, $a < b$, & y between $f(a)$ & $f(b)$ (say, $f(a) \leq f(b)$, then $f(a) \leq y \leq f(b)$), then $\exists$ $x_0 \in [a,b]$ s.t $f(x_0) = y$.



Pf Without loss of generality let $f(a) < y < f(b)$

Let
$$S = \{x \in [a,b] \mid f(x) < y\}.$$

By assumption, $a \in S$. Moreover, S is by definition bounded above by 'b'.

AOC $\implies$ $\exists$ $x \in [a,b]$ s.t $x_0 = \sup S$.

<u>Claim</u> $f(x_0) = y$.

<u>Pf</u>: $\forall$ $n \in \mathbb{N}$, $\exists$ $x_n \in S$ s.t.

$$x_0 - \frac{1}{n} < x_n \le x_0.$$

Then $x_n \longrightarrow x_0$. Also, since $f(x_n) < y$ & f is cont. $f(x_0) = \lim_{n\to\infty} f(x_n) \le y$.

Suppose $f(x_0) < y$. Let $\varepsilon = y - f(x_0) > 0$.

f cont. $\implies$ $\exists$ $\delta > 0$ s.t.

$$\left.\begin{array}{l} |x - x_0| < \delta \\ x \in [a,b] \end{array}\right\} \implies |f(x) - f(x_0)| < \varepsilon.$$

In particular

$$\left.\begin{array}{l} |x - x_0| < \delta \\ x \in [a,b] \end{array}\right\} \implies f(x) < \varepsilon + f(x_0) = y. \qquad (*)$$

$$\implies x \in S.$$

<u>CASE 1</u>. $x_0 < b$.

Pick $x \in (x_0, x_0 + \delta)$ s.t $x_0 < b$. Then

$(*) \implies x \in S$ contradicting that x_0 is $\sup S$.

CASE 2 $x_0 = b$. Then $f(x_0) = f(b) > y$.
contradicting that $f(x_0) \le y$.
So $f(x_0) = y$.

Cor 3.9 If $f: I \to \mathbb{R}$ cont. and I is an interval, then $f(I)$ is also an interval or a single point.

Example. Consider a cubic

$$p(x) = x^3 + ax + b.$$

Claim: p has at least one real root.

Pf: Note that $p(-n) \longrightarrow -\infty$ as $n \to \infty$

$p(n) \longrightarrow \infty$ as $n \to \infty$.

So $\exists\ n \in \mathbb{N}$ s.t $p(-n) < 0 < p(n)$.
Then Th^m applied to $y = 0 \implies \exists\ x \in [-n, n]$ s.t $p(x) = 0$.

· Monotonic functions

Def^n Let $f: E \to \mathbb{R}$.
① It is said to be increasing (resp. strictly increasing) if $x < y \iff f(x) \le f(y)$ (resp. $f(x) < f(y)$)
② It is said to be decreasing (resp. strictly

decreasing) if

$$x < y \iff f(y) \le f(x) \quad (\text{resp. } f(y) < f(x))$$

(3) It is said to be monotonic if it is either increasing or decreasing. Denote: $f\uparrow$ or $f\downarrow$

Thm 3.9 Let $I \subset \mathbb{R}$ be an interval & $f : I \to \mathbb{R}$ be monotonic. If $f(I)$ is also an interval, Then f is cont.

Pf: Let $p \in I$. Sps p is not an end point. (if it is, then essentially the same argument goes through with minor changes).

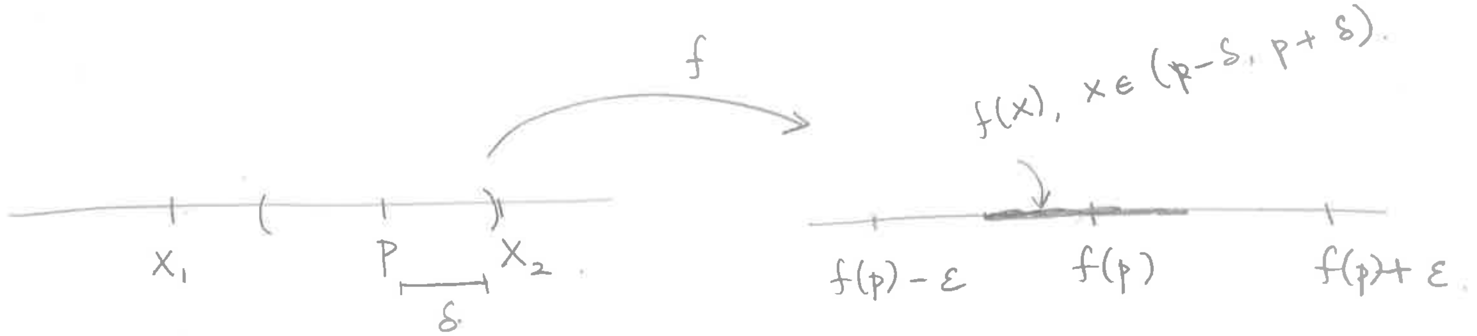
Sps also that $f\uparrow$.

Then $f(p)$ is not an end point of $f(I)$.

So $\exists\ \varepsilon_0$ s.t $(f(p)-\varepsilon_0, f(p)+\varepsilon_0) \subset f(I)$.

Let $\varepsilon > 0$. W.l.o.g sps $\varepsilon < \varepsilon_0$.

Then $(f(p)-\varepsilon, f(p)+\varepsilon) \subset f(I)$. So $\exists\ x_1, x_2 \in I$ s.t $f(x_1) = f(p) - \varepsilon$, $f(x_2) = f(p) + \varepsilon$



Since $f \uparrow$, $x_1 < p < x_2$.

Let $\delta = \min(p - x_1, x_2 - p) > 0$.

If $x \in (p-\delta, p+\delta)$, $x > x_1$ and so $f(p) - \varepsilon < f(x)$

$x < x_2$ and so $f(x) < f(p) + \varepsilon$

So $|x - p| < \delta \implies |f(x) - f(p)| < \varepsilon$.

So f is cont. at p.

Example. The Th^m i.e the converse to IVT is NOT true in general. Consider

$$f(x) = \begin{cases} \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Claim 1 f is NOT cont. at 0.

Pf: Consider $x_n = 1/n\pi$, $y_n = \dfrac{1}{2n\pi + \pi/2}$.

Then $x_n \to 0$, $y_n \to 0$. But $f(x_n) = 0$, $f(y_n) = 1$

So f is not cont. at $x = 0$.

Claim 2 f satisfies IVP.

Pf: First, note that f is continuous on $(-\infty, 0) \cup (0, \infty)$. So if a, b have same sign, then IVT $\Rightarrow$ all value between $f(a)$ & $f(b)$ are taken.

So sps now $a \leq 0 < b$. Let $n \in \mathbb{N}$ s.t $\frac{1}{2n\pi} < b$. Now,

$$f\left[\frac{1}{2(n+1)\pi}, \frac{1}{2n\pi}\right] = [-1, 1] = f(\mathbb{R}).$$

In particular, given y between $f(a)$ & $f(b)$. $\exists$ $x \in \left[\frac{1}{2(n+1)\pi}, \frac{1}{2n\pi}\right] \subset [a, b]$ s.t.

$f(x) = y$. So IVP is satisfied.

- Limits of functions.

Def^n. Let $E \subseteq \mathbb{R}$. We say $p \in \mathbb{R}$ is a limit point (l.p) of E if $\exists$ sequence $\{x_n\}$ in $E \setminus \{p\}$ s.t $x_n \to p$. Set of l.p of E is denoted by $L(E)$.

Rk: Note that p need not be in E.

Ex. 1) If $a < b$, then the set of limit points of (a, b) is $[a, b]$.

2) If $E = \mathbb{Q}$, then $L(E) = \mathbb{R}$ since any real α can be approximated by rationals not equal to α.

3) If $E = [0,2) \cup \{3\}$, Then 3 is not a l.p.

& so $L(E) = [0,2]$.

Def^n Let $E \subseteq \mathbb{R}$ and $f: E \to \mathbb{R}$. If $p \in L(E)$, we say limit of $f(x)$ as x tends to p along E is L, and write $f(x) \xrightarrow{x \to p^E} L$ or $\lim_{x \to p^E} f(x) = L$.

if $\forall\ \varepsilon > 0,\ \exists\ \delta > 0$ s.t

$$\left.\begin{array}{l} |x - p| < \delta \\ x \in E \setminus \{p\} \end{array}\right\} \implies |f(x) - L| < \varepsilon.$$

If no such L exists, we say $\lim_{x \to p^E}$ does not exist (DNE).

Rk 1) Even if $p \in E$, L might not equal $f(p)$. This is one of the reasons that in the def^n we have to take $x \in E \setminus \{p\}$.

2) When the domain $E = I \setminus \{p\}$ for some interval I, containing p, we simply write $\lim_{x \to p} f(x) = L$.

3) From the Def^n, f is cont. at $p \in E$

$\iff \lim_{x \to p^E} f(x) = f(p)$.

Th^m 3.10 Let $f: E \to \mathbb{R}$ and $p \in L(E)$. Then $\lim_{x \to p^E} f(x) = L$

$\iff \forall$ sequences $\{x_n\}$ in $E \setminus \{p\}$, s.t $x_n \to p$ we have $\lim_{n \to \infty} f(x_n) = L$.

Pf $\Longrightarrow$ Let $\varepsilon > 0$. Sps $\lim_{x \to p^E} f(x) = L$ & $\{x_n\}$ a seq^n in $E \setminus \{p\}$ s.t $x_n \longrightarrow p$.

$\exists$ $\delta > 0$ s.t

$$\begin{array}{l} |x - p| < \delta \\ x \in E \setminus \{p\} \end{array} \Longrightarrow |f(x) - L| < \varepsilon \quad (*)$$

Given $\delta > 0$, $\exists$ N s.t

$$n > N \Longrightarrow |x_n - p| < \delta.$$

$* \Longrightarrow$ if $n > N$ then $|f(x_n) - L| < \varepsilon$.

So $\lim_{n \to \infty} f(x_n) = L$.

$\Longleftarrow$ Sps not. Then $\exists$ $\varepsilon > 0$ s.t $\forall$ $n \in \mathbb{N}$, $\exists$ $x_n \in E \setminus \{p\}$ s.t

$$|x_n - p| < \frac{1}{n} \quad \text{but} \quad |f(x_n) - L| \geq \varepsilon. \quad (**)$$

Now $x_n \longrightarrow p$. Hypothesis $\Longrightarrow$ $\lim_{n \to \infty} f(x_n) = L$

i.e $\exists$ $N > 0$ s.t $|f(x_N) - L| < \varepsilon$. contradicting $(**)$.

Cor 3.11 If $\lim_{x \to p^E} f(x) = L_1$, $\lim_{x \to p^E} f(x) = L_2$ Then $L_1 = L_2$.

Examples. 1) $f(x) = \sin\left(\frac{1}{x}\right)$, $x \in \mathbb{R}\setminus\{0\} = E$

Claim: $\lim_{x\to 0} f(x)$ DNE.

Pf: Consider $x_n = \frac{2}{(2n+1)\pi} \xrightarrow{n\to\infty} 0$

Then $f(x_n) = (-1)^n \quad \forall n$.

Since $f(x_n)$ does not converge.

Th^m $\Longrightarrow$ $\lim_{x\to 0} f(x)$ DNE

2) $g(x) = x\sin\left(\frac{1}{x}\right)$, $x \in \mathbb{R}\setminus\{0\}$.

Claim $\lim_{x\to 0} g(x) = 0$.

Pf Let $\varepsilon > 0$. Note

$$|g(x)| = \left|x \sin\left(\frac{1}{x}\right)\right| \le |x|.$$

So if $\delta = \varepsilon$, Then

$$\begin{matrix}|x| < \delta \\ x \neq 0\end{matrix} \Longrightarrow |g(x)| \le |x| < \delta = \varepsilon.$$

So $\lim_{x\to 0} g(x) = 0$.

Cor 3.12 Let $f_1, f_2 : E \to \mathbb{R}$ with limits $\lim_{x\to p^E} f_1(x) = L_1$, $\lim_{x\to p^E} f_2(x) = L_2$. Then

(1) $\lim_{x\to p^E} f_1 + f_2 = L_1 + L_2$.

(2) $\lim_{x\to p^E} f_1 \cdot f_2 = L_1 \cdot L_2$.

③ $\lim_{x \to p^E} \frac{f_1}{f_2} = \frac{L_1}{L_2}$ if $L_2 \neq 0$ and $f_2(x) \neq 0$ in $E \cap I$ for some interval I containing p.

<u>Th^m 3.13</u> (Compositions) Let $f: E \to \mathbb{R}$ & $p \in L(E)$ s.t $\lim_{x \to p^E} f(x) = L$. If g is defined on $f(E) \cup \{L\}$, and is <u>continuous</u> at L, then

$$\lim_{x \to p^E} g \circ f(x) = g(L).$$

<u>Rk</u>: 1) In common examples $g: \mathbb{R} \to \mathbb{R}$ is cont., so Theorem applies.

e.g. $g(x) = e^x$. Then $\lim_{x \to p} e^{f(x)} = e^{\lim_{x \to p} f(x)}$.

2) Continuity of g is required in Th^m.

Consider $f(x) = 1 + x \sin\left(\frac{\pi}{x}\right)$, $x \in \mathbb{R} \setminus \{0\} = E$.

$$g(x) = \begin{cases} 4, & x \neq 1 \\ -4, & x = 1. \end{cases}$$

Clearly $\lim_{x \to 0} f(x) = 1$ & $\lim_{x \to 1} g(x) = 4$.

Now let $x_n = \frac{2}{n}$. Then $f(x_n) = \begin{cases} 1, & n \text{ is even} \\ \neq 1, & n \text{ is odd}. \end{cases}$

$$\Rightarrow g \circ f(x_n) = \begin{cases} -4, & n \text{ is even} \\ 4, & n \text{ is odd}. \end{cases}$$

So $\lim_{x \to 0} g \circ f(x)$ DNE.

<u>Pf of Th^m</u> Note $g \circ f$ is defined on E. Let $x_n \in E \setminus \{p\}$ s.t $x_n \to p$. Then $f(x_n) \to L$. Since g is cont.

$$g \circ f(x_n) = g(f(x_n)) \longrightarrow g(L).$$

Since this works for all sequences, $\lim_{x \to p} g \circ f = g(L)$

• <u>One sided limits</u>

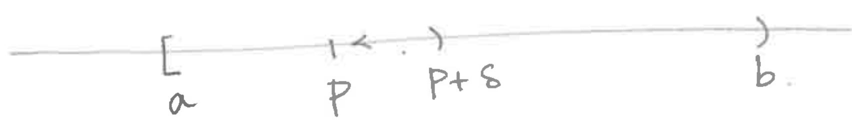
<u>Def^n</u>: Let $f : (a, b) \longrightarrow \mathbb{R}$.

1) (Right hand limit). For $p \in [a, b)$, we say that the limit of $f(x)$ as x tends to p from the right, and write

$$f(p+) = \lim_{x \to p^+} f(x) = L$$

if $\forall \ \varepsilon > 0, \ \exists \ \delta > 0$ s.t

$$x \in (p, p+\delta) \implies |f(x) - L| < \varepsilon.$$



2) (Left hand limit) Similarly, for $p \in (a, b]$ we define $f(p-) = \lim_{x \to p^-} f(x) = L$ if $\forall \ \varepsilon > 0, \exists \ \delta >$ s.t

$$x \in (p-\delta, p) \implies |f(x) - L| < \varepsilon.$$

Th^m: 3.14 $\lim_{x\to p} f(x) = L \iff f(p+) = f(p-) = L$.

Example Let $f(x) = \dfrac{|x-2|}{x^2-4} = \begin{cases} \dfrac{1}{x+2} & x \geq 2 \\ \dfrac{-1}{x+2}, & x < 2 \end{cases}$

So $f(2+) = \dfrac{1}{4}$

$f(2-) = \dfrac{-1}{4}$.

Since $f(2+) \neq f(2-)$, $\lim_{x\to 2} f(x)$ DNE.

- Limits to $\pm\infty$ and infinite limits

Def^n: Let $f: E \to \mathbb{R}$ & $p \in L(E)$.

1) We say $\lim_{x\to p^E} f(x) = \infty$ if $\forall M \ \exists \ \delta > 0$ s.t

$$\begin{rcases} |x-p| < \delta \\ x \in E\setminus\{p\} \end{rcases} \Longrightarrow f(x) > M.$$

2) We say $\lim_{x\to p^E} f(x) = -\infty$ if $\forall M, \exists \ \delta > 0$ s.t

$$\begin{rcases} |x-p| < \delta \\ x \in E\setminus\{p\} \end{rcases} \Longrightarrow f(x) < -M.$$

Similarly we can define $f(p+), f(p-) = \pm\infty$.

Example $\lim_{x \to p} \frac{1}{|x-p|} = \infty$.

$\lim_{x \to p} \frac{1}{x-p}$ DNE.

In fact $\lim_{x \to p^+} \frac{1}{x-p} = +\infty$

$\lim_{x \to p^-} \frac{1}{x-p} = -\infty$.

Defn: 1) Let $f : (a, \infty) \to \mathbb{R}$. We say $\lim_{x \to \infty} f(x) = L$ if $\forall \, \varepsilon > 0$, $\exists \, M$ s.t

$x > M \implies |f(x) - L| < \varepsilon$.

Similarly we define $\lim_{x \to \infty} f(x) = \pm \infty$.

2) Let $f : (-\infty, a) \to \mathbb{R}$. We say $\lim_{x \to -\infty} f(x) = L$ if $\forall \, \varepsilon > 0$, $\exists \, M$ s.t

$x < -M \implies |f(x) - L| < \varepsilon$.

Similarly define $\lim_{x \to -\infty} f(x) = \pm \infty$.

- Classification of discontinuities

Def^n. Let $f : (a,b) \longrightarrow \mathbb{R}$ and $p \in (a,b)$.

1) We say that f has a removable discont. at p if $f(p+)$ & $f(p-)$ exist and
$$f(p+) = f(p-) \neq f(p).$$

2) We say f has a jump discont. at p if $f(p+)$ & $f(p-)$ exist &
$$f(p+) \neq f(p-).$$

3) We say f has an essential discont. at p if at least one of $f(p+)$ & $f(p-)$ do not exist.

Rk: In case of 1), we can re-define f by

$$\tilde{f}(x) = \begin{cases} f(x), & x \neq p \\ f(p+) = f(p-), & x = p \end{cases}$$

Then $\tilde{f}$ is cont. at p, hence the name removable discont.

Examples 1) $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$

Neither $f(p+)$ or $f(p-)$ exist for any $p \in \mathbb{R}$.
So f has an essential dis cont. at all $p \in \mathbb{R}$.

② $$f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

f is cont. at $x = 0$, but essential dis cont. at all $p \in \mathbb{R} \setminus \{0\}$.

③ $$f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 1+x, & 1 < x < 2. \end{cases}$$

Clearly f is cont at all $x \neq 1$ i.e on $[0,1) \cup (1,2)$

<u>At $x=1$</u>:

$$f(1+) = \lim_{x \to 1^+} 1 + x = 2.$$

$$f(1-) = \lim_{x \to 1^-} x = 1.$$

So f has a jump dis cont. at $x = 1$.

There is a class of functions with only jump discontinuities.

$f : (a, b) \to \mathbb{R}$.

Thm 3.15 A monotonic function, has <u>no</u> essential discontinuity i.e $f(p-)$ & $f(p+)$ always exist.

Rk 1) This is analogous to the MCT.

2) $f \uparrow \implies f(p-) \leq f(p+)$

$f \downarrow \implies f(p-) \geq f(p+)$.

Pf of Thm Sps $f \uparrow$ (other case is similar).

Let $p \in (a, b)$.

Set $A = \sup_{t \in (a,p)} f(t)$, $B = \inf_{t \in (p,b)} f(t)$.

Claim $f(p-) = A$, $f(p+) = B$.

Pf: Note that $f \uparrow \implies A \leq B$ and $A \neq \infty$, $B \neq -\infty$

Let $\varepsilon > 0$. $A = \sup_{t \in (a,p)} f(t) \implies \exists\ \delta > 0$ s.t

$$A - \varepsilon < f(p - \delta) \leq A$$

$f \uparrow \Rightarrow \forall\ x \in (p-\delta, p)$

$$A - \varepsilon < f(p-\delta) \le f(x) \le A$$

So given $\varepsilon > 0$, $\exists$ $\delta > 0$ s.t

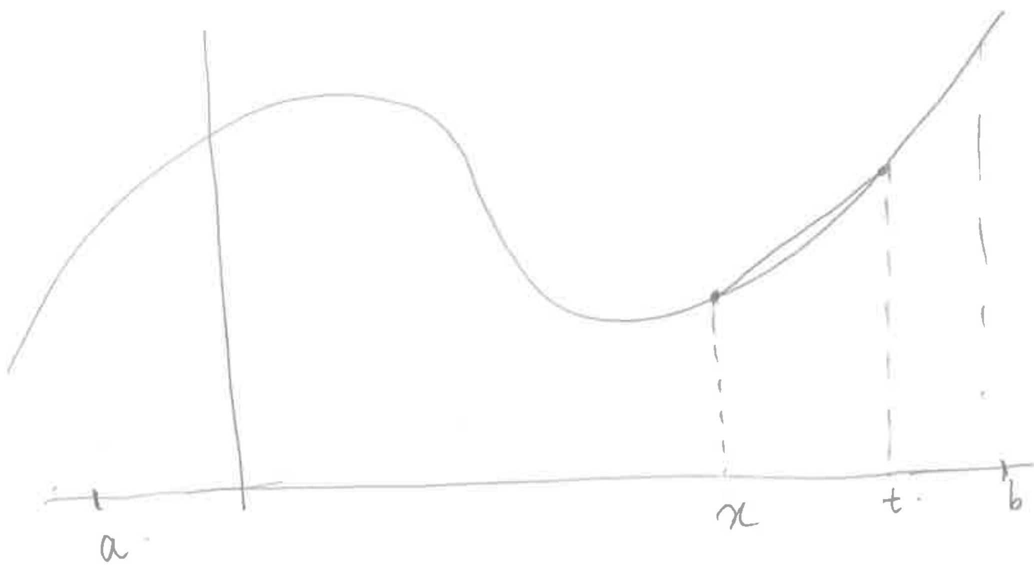
$$x \in (p-\delta, p) \Rightarrow |f(x) - A| < \varepsilon.$$

$\Rightarrow f(p-)$ exists & $f(p-) = A$

Similarly $f(p+)$ exists & $f(p+) = B$.

Chapter 4 - Differentiation

• <u>The derivative</u>. Let $f : (a,b) \longrightarrow \mathbb{R}$ & $p \in (a,b)$.



<u>Def^n</u> The <u>difference quotient</u> of f at p w.r.t 't' is defined to be

$$\varphi(t) := \frac{f(t) - f(x)}{t - x}, \quad t \neq p$$

<u>Rk</u> $\varphi(t)$ is simply the <u>slope</u> of the secant line joining $(p, f(p))$ to $(t, f(t))$.

<u>Def^n</u> We say f is <u>differentiable</u> at 'p' if

$$f'(p) := \lim_{t \to p} \varphi(t).$$

exists. We then call $f'(p)$ the derivative of f at p.

Rk Geometrically, $f'(x)$ is the slope of the tangent line to $y = f(x)$ at $(p, f(p))$

Notation: $f'(p) = \left.\dfrac{df}{dx}\right|_{x=p}$

Def^n We say f is diff. on (a,b) if it is differentiable at all points $x \in (a,b)$. Then $f'(x)$ itself is a function $f' : (a,b) \to \mathbb{R}$.

Examples 1) Constants: $f(x) = c \quad \forall \; x \in (a,b)$

Then $\varphi(t) = 0 \quad \forall \; t \neq x$ & so

$$f'(x) = 0 \quad \forall \; x \in (a,b).$$

2) Monomials: $f(x) = x^n$, $n \in \mathbb{N}$.

Now if $t \neq x$ $\varphi(t) = \dfrac{f(t) - f(x)}{t - x} = \dfrac{t^n - x^n}{t - x}$

$$= t^{n-1} + t^{n-2}x + \cdots + tx^{n-2} + x^{n-1}$$

$$\xrightarrow{t \to x} \underbrace{x^{n-1} + x^{n-1} + \cdots + x^{n-1}}_{n-\text{terms}}$$

$$= n x^{n-1}$$

So, x^n is diff on $\mathbb{R}$ and

$$\boxed{\frac{d}{dx} x^n = n x^{n-1}}$$

In fact we have the foll.

Thm 4.1 Let $\alpha \in \mathbb{R}$. Then x^α is differentiable on $(0, \infty)$ and

$$\boxed{\frac{d}{dx} x^\alpha = \alpha x^{\alpha - 1}}$$

We'll give a proof later.

3) Trig functions We'll prove later that $\sin(x)$ & $\cos(x)$ are diff on $\mathbb{R}$ and

$$\frac{d}{dx} \sin(x) = \cos x$$

$$\frac{d}{dx} \cos x = -\sin(x).$$

4) Exp. Once we define e^x, we will show it is diff on $\mathbb{R}$ &

$$\frac{d}{dx} e^x = e^x.$$

5) Absolute Value $\quad f(x) = |x|$ on $\mathbb{R}$.

$$= \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases}$$

Clearly f is diff for $x \neq 0$. In fact

$$f'(x) = \begin{cases} 1, & x = 0 \\ -1, & x \neq 0. \end{cases}$$

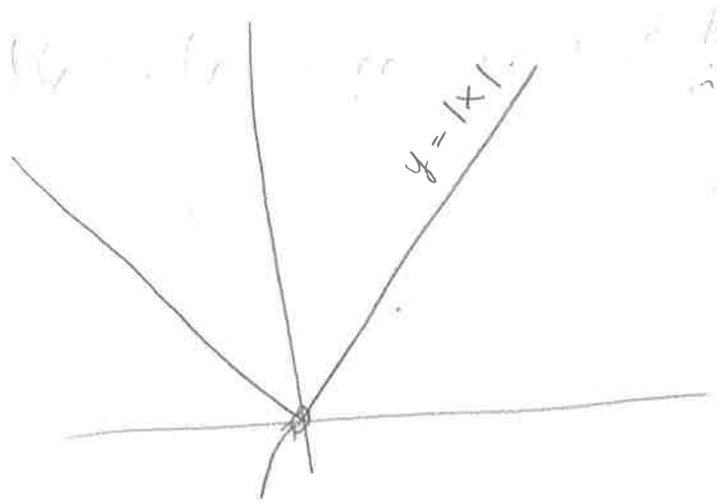
At $x = 0$.

$$\varphi(t) = \frac{f(t) - f(0)}{t} = \begin{cases} 1, & t \geq 0 \\ -1, & t < 0 \end{cases}$$

So $\varphi(0+) = 1$, $\varphi(0-) = -1$.

& $\lim_{t \to 0} \varphi(t)$ DNE.

So f is NOT differentiable at $x = 0$



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