WEEK - 2

Day - 1

Chapter - 2 Sequences & Series (cont.)

- Recall: We ended with the foll. theorem (Bolzano - Weierstrass). Every bounded seq^n in $\mathbb{R}$ has a convergent sub-seq^n.

- Cauchy Criteria.

Def^n: A seq^n $\{a_n\}$ in $\mathbb{R}$ is called Cauchy if $\forall\ \varepsilon > 0$, $\exists\ N \in \mathbb{N}$ s.t

$$n, m > N \implies |a_n - a_m| < \varepsilon.$$

i.e eventually all terms are close by.

Example: $a_n = (-1)^n / n^2$.

Claim: $\{a_n\}$ is Cauchy.

Pf. Note $|a_n - a_m| = \left| \frac{(-1)^n}{n^2} - \frac{(-1)^m}{m^2} \right| \le \frac{1}{n^2} + \frac{1}{m^2}$

by Δ-ineq. Since $1/n^2 \to 0$, $1/m^2 \to 0$ as $m, n \to \infty$, given $\varepsilon > 0$, $\exists\ N$ s.t

$$m, n > N \implies \frac{1}{n^2}, \frac{1}{m^2} < \frac{\varepsilon}{2}$$

Then

$$m, n > N \implies |a_n - a_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Note that $\{a_n\}$ also converges.

<u>Th^m 2.12</u> (Cauchy criteria). A seq $\{a_n\}$ is convergent if and only if $\{a_n\}$ is Cauchy.

<u>Rk</u> The advantage of this Th^m is that one can show convergence <u>without</u> knowing the limit.

First we need the foll.

<u>Lemma 2.13</u> Cauchy sequences are bounded.

<u>Pf</u>: Sps $\{a_n\}$ is Cauchy. Then taking $\varepsilon = 1$, $\exists$ N s.t

$$m, n > N \implies |a_n - a_m| < 1.$$

In particular $|a_n - a_{N+1}| < 1$.

$$\triangle\text{-ineq} \implies |a_n| \leq |a_n - a_{N+1}| + |a_{N+1}| < |a_{N+1}| + 1.$$

Let $M = \max(|a_1|, |a_2|, \cdots, |a_N|, |a_{N+1}| + 1)$.

Then $|a_n| \leq M \quad \forall \; n \in \mathbb{N}$.

So $\{a_n\}$ is bounded.

<u>Pf of Th^m 2.12</u>.

$\implies$ Sps $\{a_n\}$ is convergent, say $a_n \to L$.

Given $\varepsilon > 0$, $\exists$ N s.t

$$n > N \implies |a_n - L| < \frac{\varepsilon}{2}.$$

Then by Δ-ineq

$$m, n > N \implies |a_m - a_n| = |a_m - L + L - a_n|$$

$$\leq |a_m - L| + |a_n - L|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So $\{a_n\}$ is Cauchy.

$\underline{\Longleftarrow}$ Now, sps $\{a_n\}$ is Cauchy. Then Lemma 2.13 $\implies \{a_n\}$ is bounded.

Bolzano-Weierstrass $\implies \exists$ conv. sub-seq^n, say $a_{n_k} \longrightarrow L$.

<u>Claim</u>. $a_n \longrightarrow L$.

<u>Pf</u>: $a_{n_k} \longrightarrow L \implies$ Given $\varepsilon > 0$, $\exists$ $K > 0$ s.t

$$k > K \implies |a_{n_k} - L| < \varepsilon/2. \qquad (*)$$

$\{a_n\}$ Cauchy $\implies \exists$ N s.t.

$$m, n > N \implies |a_n - a_m| < \varepsilon/2. \qquad (**)$$

Choose $k_0 > K$ s.t $n_{k_0} > N$. Then

$$n > N \implies |a_n - L| \leq |a_n - a_{n_{k_0}}| + |a_{n_{k_0}} - L|$$

$$< \overset{(**)}{\frac{\varepsilon}{2}} + \overset{(*)}{\frac{\varepsilon}{2}}.$$

$$= \varepsilon.$$

So $a_n \longrightarrow L$.

The proof yields the foll useful result.

Cor 2.12 If $\{a_n\}$ is Cauchy and has a subseq^n $a_{n_k} \longrightarrow L$. Then $a_n \longrightarrow L$.

Rk: We have various form of the completeness of $\mathbb{R}$ namely axiom of completeness (AOC), monotone convergence theorem (MCT), Bolzano-Weierstrass (BW) and Cauchy criteria (CC). None of them are true for $\mathbb{Q}$. We have seen

$$AOC \implies MCT \implies BW \implies CC.$$

Would any of the other three serve equally well as AOC as the starting point?

Fact: $AOC \iff MCT \iff BW$.

But: $CC \not\Rightarrow AOC$.

The problem is, starting only from CC, there is no way to prove that $\lim_{n\to\infty} \frac{1}{n} = 0$. That is a consequence of Archimedean property.

Fact $CC + \text{Archimedean} \iff AOC$.

• Limit Superior and inferior

Let $\{a_n\}$ be upper bounded. Then if

$$u_n = \sup\{a_k \mid k > n\}, \quad \leftarrow \text{tail of seq}^n.$$

we have $u_{n+1} \le u_n$ (since sup is taken over smaller set), so $u_n \downarrow$.

Similarly if $\{a_n\}$ lower bounded, define

$$l_n = \inf\{a_k \mid k > n\}.$$

Then $l_n \uparrow$

Def^n 1) The limit superior of a seq^n $\{a_n\}$ is defined to be

$$\limsup_{n\to\infty} a_n := \lim_{n\to\infty} u_n = \lim_{n\to\infty} \sup\{a_k \mid k > n\}.$$

if $\{a_n\}$ bounded above, or ∞ otherwise.

2) The limit inferior of a seq^n $\{a_n\}$ is defined to be

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} l_n = \lim_{n\to\infty} \inf\{a_k \mid k > n\}.$$

if $\{a_n\}$ is bounded below, or $-\infty$ otherwise.

Examples: 1) $a_n = -n$.

Then $u_n = \sup\{a_k \mid k > n\} = -n \to -\infty$.

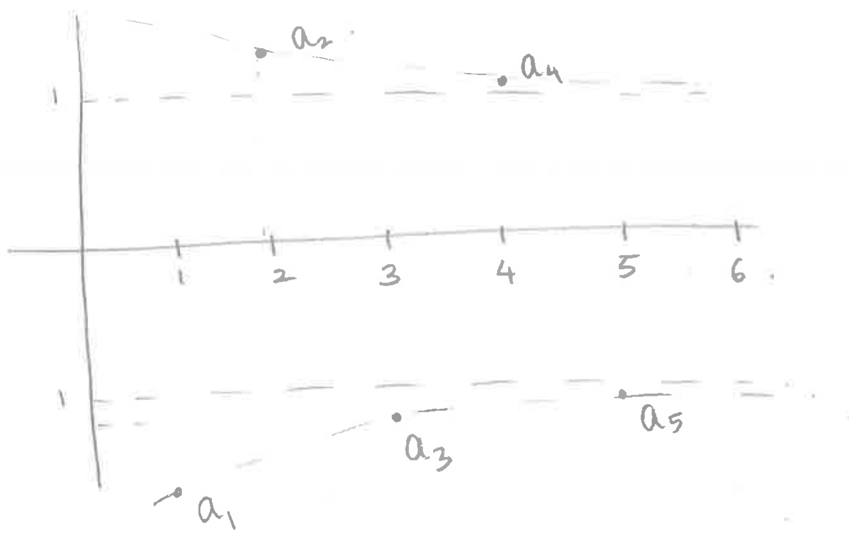
So $\limsup_{n\to\infty} a_n = -\infty$.

a_n not bounded below, so $\liminf_{n\to\infty} a_n = -\infty$.

2) $a_n = (-1)^n \cdot \frac{n+1}{n}$.

$u_n = \sup\left\{(-1)^k \cdot \frac{k+1}{k} \mid k > n\right\}$.

Clearly $u_2 = a_2$, $u_3 = u_4 = a_4$, $u_5 = u_6 = a_6$ .....



So $\lim_{n\to\infty} u_n = \lim_{n\to\infty} a_{2n} = 1$.

So $\limsup_{n\to\infty} a_n = 1$

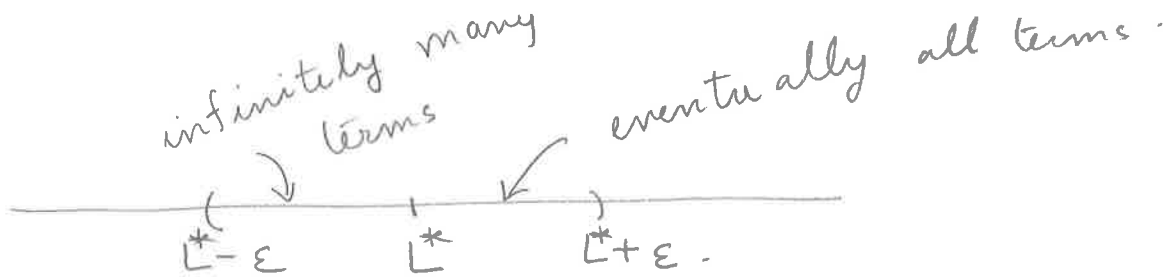
Similarly $\liminf_{n\to\infty} a_n = -1$

Thm 2.15 Let $\{a_n\}$ be a sequence bounded above. The foll. are equivalent.

(1) $L^* = \limsup_{n\to\infty} a_n \in \mathbb{R}$ (and not $\mathbb{R}^*$).

(2) $\forall\, \varepsilon > 0$ (a). $\exists\, N$ s.t $n > N \Rightarrow a_n < L^* + \varepsilon$.

(b) $\forall\, N$, $\exists\, n > N$ s.t $a_n > L^* - \varepsilon$.



Pf: Let $u_n = \sup\{a_k \mid k > n\}$ as before.

$\Longrightarrow$: Sps $L^* = \limsup_{n\to\infty} a_n \in \mathbb{R}$.

$u_n \searrow L^*$. So $L^* = \inf u_n$.

$\Rightarrow \exists\, N$ s.t $u_N < L^* + \varepsilon$.

$\Rightarrow \forall\, n > N$, $a_n < L^* + \varepsilon$ ---- verified (a).

Also, $L^* = \inf u_n \Rightarrow u_N > L^* - \varepsilon \;\; \forall\, N$.

But then $\forall\, N$, $\exists\, n > N$ s.t $\boxed{a_n > L^* - \varepsilon}$.

or else $L^* - \varepsilon$ would be an u.b for $\{a_k \mid k > N\}$ contradicting u_N is l.u.b.

So (b) is verified

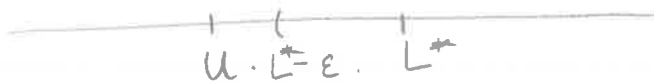
$\Leftarrow$ Sps L^* satisfies (a), (b). Let

$$U = \limsup_{n\to\infty} a_n .$$

Claim. $U = L^*$.

Pf. Sps not. Then

CASE 1 $\quad U < L^*$.



Choose $\varepsilon > 0$ s.t $\quad U < L^* - \varepsilon < L^*$.

(b) $\Rightarrow \quad \forall N, \ \exists \ n > N$ s.t $\quad a_n > L - \varepsilon$.

$\Rightarrow \quad U_n > L^* - \varepsilon \quad \forall n$.

$\Rightarrow \quad U = \lim_{n\to\infty} U_n > L^* - \varepsilon \quad$ contradiction!

CASE 2 $\quad L < U$.

Choose $\varepsilon > 0$ s.t. $\quad L^* < L^* + \varepsilon < U$.

(a) $\Rightarrow \ \exists \ N$ s.t $\ \forall \ n > N, \quad a_n < L^* + \varepsilon$.

$\Rightarrow \forall n > N, \ \sup\{a_k \mid k > n\} < L^* + \varepsilon$.

$\Rightarrow \forall n > N, \quad U_n < L^* + \varepsilon$.

$\Rightarrow \quad U = \lim_{n\to\infty} U_n < L^* + \varepsilon$. $\quad$ Contradiction

So $\quad U = L^*$.

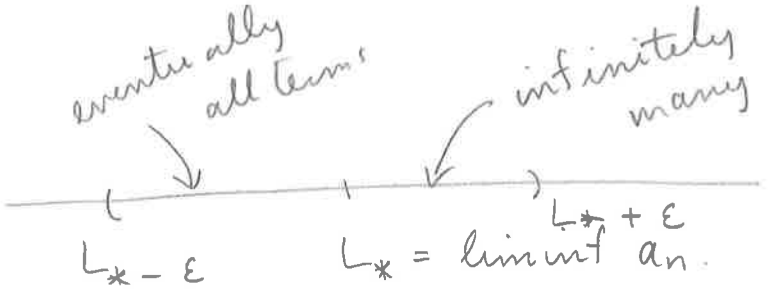
Similarly we have

<u>Th^m 2.16.</u> Let $\{a_n\}$ be a seq^n bounded below. TFAE

(1) $L_* = \liminf\limits_{n\to\infty} a_n \in \mathbb{R}$.

(2) $\forall\, \varepsilon > 0$. (a) $\exists\ N$ s.t $n > N \Longrightarrow a_n > L_* - \varepsilon$.

(b) $\forall\, N$, $\exists\, n > N$ s.t $a_n < L_* - \varepsilon$.



<u>Cor 2.17</u> A seq^n $\{a_n\}$ converges to L

$\Longleftrightarrow \limsup\limits_{n\to\infty} a_n = \liminf\limits_{n\to\infty} a_n = L$.

<u>Pf</u> <u>$\Longrightarrow$</u>. Sps $\lim\limits_{n\to\infty} a_n = L$.

$\forall\, \varepsilon > 0$, $\exists\ N$ s.t.

$$L - \varepsilon < a_n < L + \varepsilon.$$

$$\Longrightarrow \quad L - \varepsilon < u_n, l_n < L + \varepsilon$$

Since $\limsup a_n = \lim\limits_{n\to\infty} u_n$

$$\Longrightarrow \quad L - \varepsilon < \limsup_{n\to\infty} a_n < L + \varepsilon$$

$\forall\, \varepsilon > 0 \quad \Longrightarrow \quad \limsup\limits_{n\to\infty} a_n = L$.

Similarly $\liminf_{n\to\infty} a_n = L$.

$\Leftarrow$ Let $\varepsilon > 0$.

$L = \limsup_{n\to\infty} a_n \Longrightarrow \exists\, N_1$ s.t $\forall\, n > N_1$

$a_n < L + \varepsilon$.

$L = \liminf_{n\to\infty} a_n \Longrightarrow \exists\, N_2$ s.t $\forall\, n > N_2$

$a_n > L - \varepsilon$.

If $N = \max(N_1, N_2)$

$n > N \Longrightarrow L - \varepsilon < a_n < L + \varepsilon$

or $|a_n - L| < \varepsilon$.

So $a_n \longrightarrow L$.

Rk Note that $\liminf_{n\to\infty} a_n \le \limsup_{n\to\infty} a_n$ always holds.

Day - 2

• Infinite Series

Summation notation : The notation

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \cdots + a_n .$$

The index 'k' is unimportant i.e $\sum_{k=n}^{m} a_k = \sum_{j=n}^{m} a_j$

Ex. $$\sum_{k=2}^{5} \frac{1}{k+k^2} = \frac{1}{2^2+2} + \frac{1}{3^2+3} + \frac{1}{4^2+4} + \frac{1}{5^2+5}$$
$$= \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} .$$

Defn. Let $m \in \mathbb{N}$, and $\{a_n\}_{n=m}^{\infty}$ be a seqn. We define the seqn $\{s_n\}_{n=m}^{\infty}$ of partial sums by

$$s_n = \sum_{k=m}^{n} a_k = a_m + a_{m+1} + \cdots + a_n$$

Typically $m = 0$ or 1.

Defn We say that $\sum a_n$ converges if the seqn $\{s_n\}$ converges, and we define the sum of the infinite series $\sum a_n$ by

$$S = \sum_{k=m}^{\infty} a_k := \lim_{n\to\infty} s_n .$$

If $\{s_n\}$ diverges, we say $\sum a_n$ diverges.

If $\{s_n\}$ diverges to $\pm\infty$, we say $\sum_{n=m}^{\infty} a_n = \pm\infty$.

Examples 1) Geometric Series A geometric series is a series of the form

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots$$

n^{th} partial sum: $s_n = \sum_{k=0}^{n} ar^k = a \sum_{k=0}^{n} r^k$.

Then $$r s_n = a \sum_{k=0}^{n} r^{k+1}$$

$$= a\left[r^{n+1} - 1 + \sum_{k=0}^{n} r^k\right]$$

$$= a(r^{n+1} - 1) + s_n.$$

So $$s_n = \frac{a(1 - r^{n+1})}{1 - r} \quad \text{if } r \neq 1.$$

i.e. $$\boxed{a(1 + r + r^2 + \cdots + r^n) = a\,\frac{1 - r^{n+1}}{1 - r}} \quad \text{if } r \neq 1.$$

As $n \to \infty$,

$|r| < 1$ then $r^{n+1} \to 0$, so $s_n \to \dfrac{a}{1-r}$

$|r| > 1$ then r^{n+1} diverges, so s_n diverges

$r = 1$, $s_n = na \longrightarrow \pm\infty$

Th^m 2.18 The series $\sum a r^n$ converges if and only if $|r| < 1$. Then

$$\sum_{n=0}^{\infty} a r^n = \frac{a}{1-r}.$$

For instance if $r = 2^{-1}$, $a = 1$, $\sum_{n=0}^{\infty} \frac{1}{2^n} = 1$.

2) **P- Series** $\sum_{n=1}^{\infty} n^{-p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$

$p < 0$: Then each $n^{-p} \geq 1$ if $n \geq 1$.

So $s_n = \sum_{k=1}^{n} k^{-p} \geq n$.

Since $s_n \uparrow$ (adding +ve terms)

$\Rightarrow s_n \to +\infty$ as $n \to \infty$.

So series diverges.

$p = 1$ $s_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}} + \cdots$

We group terms. $\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}.$$

In general

$$\frac{1}{2^{k-1}+1} + \frac{1}{2^{k-1}+2} + \frac{1}{2^{k-1}+3} + \cdots + \frac{1}{2^k}$$

$$> \frac{2^k - 2^{k-1}}{2^k} = \frac{2^{k-1}}{2^k} = \frac{1}{2}.$$

So if $n = 2^m$,

$$s_n \geqslant 1 + \frac{1}{2} + \frac{1}{2} + \cdots \frac{1}{2}.$$

$$= 1 + \frac{m}{2}.$$

As $m \to \infty$ $s_{2^m} \to \infty$. i.e. $\{s_n\}$ is <u>not bounded</u>

So s_n diverges $\Longrightarrow \sum \frac{1}{n}$ diverges

<u>$p = 2$</u>. $s_n = \sum_{k=1}^{n} \frac{1}{k^2} = 1 + \underbrace{\frac{1}{2^2} + \frac{1}{3^2}} + \underbrace{\frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2}}$

$+ \underbrace{\frac{1}{8^2} + \cdots}$

Now $\frac{1}{2^2} + \frac{1}{3^2} < \frac{1}{2^2} + \frac{1}{2^2} = \frac{1}{2}$

$\frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} < \frac{1}{4^2} + \cdots + \frac{1}{4^2} = \frac{4}{4^2} = \frac{1}{4}.$

$\frac{1}{8^2} + \cdots + \frac{1}{15^2} < \frac{8}{8^2} = \frac{1}{8} = \frac{1}{2^3}.$

In general

$$\frac{1}{(2^k)^2} + \frac{1}{(2^k+1)^2} + \cdots + \frac{1}{(2^{k+1}-1)^2} < \frac{2^k}{2^{2k}} = \frac{1}{2^k}.$$

So if $n = 2^m - 1$,

$$s_{2^m-1} < 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots \frac{1}{2^m} < \sum_{m=0}^{\infty} \frac{1}{2^m} = 1.$$

So $\quad s_{2^m-1} < 1.$

Since for any n, $\exists\ m$ s.t $n < 2^m - 1$,

$$s_n < 1 \quad \forall\ n.$$

So $s_n \uparrow$ and bounded $\Longrightarrow s_n$ converges.

$\Longrightarrow \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

Using a similar argument one can show.

__Th^m 2.19__ (p- series convergence) $\sum_{n=1}^{\infty} n^{-p}$ converges if and only if $p > 1$.

__Rk__: We'll later prove this using the integral test

3) __Telescoping series__

__e.g__: Consider $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots$

$$= \sum_{k=1}^{\infty} \frac{1}{k(k+1)}.$$

Then

$$s_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{(n-1)n} + \frac{1}{n(n+1)}.$$

$$= \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right)$$

$$= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots \frac{1}{n-1} - \frac{1}{n} + \frac{1}{n} - \frac{1}{n+1}$$

$$= 1 - \frac{1}{n+1}. \longrightarrow 1 \text{ as } n \to \infty.$$

So $\displaystyle\sum_{n=1}^{\infty} \frac{1}{k(k+1)} = 1.$

In general.

Th^m 2.20 Let $\{a_n\}$ seq^n s.t $a_n = b_n - b_{n+1}$ for some seq^n $\{b_n\}$. Then $\sum a_n$ converges $\iff \lim_{n\to\infty} b_n = 0$, and then $\displaystyle\sum_{n=1}^{\infty} a_n = b_1$.

- Some general theorems con

Th^m 2.21 (Cauchy Criteria). $\sum a_n$ converges $\iff \forall\, \varepsilon > 0,\ \exists\, N$ s.t

$$n \geq m \geq N \implies \left|\sum_{k=m}^{n} a_k\right| < \varepsilon.$$

Cauchy criteria for series

called the tail

Pf $\sum a_n$ conv. $\Longleftrightarrow$ $\{s_n\}$ is Cauchy.

$\Longleftrightarrow$ $\forall \varepsilon > 0$, $\exists$ N_1 s.t

$n > l > N_1$, then $|s_n - s_l| < \varepsilon$.

$\Longleftrightarrow$ $\forall \varepsilon > 0$, $\exists$ N_1 s.t

$n > l > N_1$, then $\left| \sum_{k=l+1}^{n} a_k \right| < \varepsilon$.

$\underset{N_1+1=N}{\overset{l+1=m}{\Longleftrightarrow}}$ $\forall \varepsilon > 0$, $\exists$ N s.t

$n \geq m > N$, $\left| \sum_{k=m}^{n} a_k \right| < \varepsilon$.

Cor 2.22 (Divergence test). If $\sum a_n$ conv.

then $\lim_{n\to\infty} a_n = 0$

Pf: Apply CC for series with $n = m$.

$\sum a_n$ conv. $\Longrightarrow$ $\forall \varepsilon > 0$, $\exists$ N s.t

$$\left| \sum_{k=n}^{n} a_k \right| = |a_n| < \varepsilon.$$

$\Longrightarrow$ $\lim_{n\to\infty} a_n = 0$.

Rk 1) Converse NOT true. eg. $\sum n^{-1}$ Then

$\lim_{n\to\infty} n^{-1} = 0$ but $\sum n^{-1}$ diverges.

2) Usually applied to prove divergence. Given

$\sum a_n$, first thing to check is if $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges!

Ex. Consider $\sum n^{1/n}$. Then $\lim_{n\to\infty} n^{1/n} = 1$, so

$\sum n^{1/n}$ diverges

Th^m 2.23 (Basic alg. properties). If $\sum_{n=1}^{\infty} a_n = A$, $\sum_{n=1}^{\infty} b_n = B$. Then

(1) For any $c \in \mathbb{R}$ $\sum c a_n$ converges &

$$\sum_{n=1}^{\infty} c a_n = cA.$$

(2) $\sum (a_n + b_n)$ conv. and

$$\sum_{n=1}^{\infty} (a_n + b_n) = A + B.$$

• Series with non-negative terms

Th^m 2.24 Let $a_n \geq 0$. Then $\sum a_n$ conv. $\iff \{s_n\}$ bounded.

Pf: $a_n \geq 0 \implies s_n \uparrow$. So $\{s_n\}$ conv $\iff \{s_n\}$ bound.

Th^m 2.25 (Comparison test). Let $0 \leq a_n \leq b_n \ \forall\ n > m$.

(1) $\sum b_n$ conv. $\implies \sum a_n$ conv.

(2) $\sum a_n$ div $\implies \sum b_n$ div

For proof we will take $m=1$. since the initial few terms do not change conv./div

Pf Let $s_n = \sum_{k=1}^{n} a_k$, $t_n = \sum_{k=1}^{n} b_k$, Note $s_n \le t_n$.

(1) $\sum b_n$ conv $\Longrightarrow$ $\{t_n\}$ is bounded

$\Longrightarrow$ $\{s_n\}$ is bounded

$\Longrightarrow$ $\sum a_n$ conv.

(2) $\sum a_n$ div. $\Longrightarrow$ $s_n \to \infty$

$\Longrightarrow$ $t_n \to \infty$

$\Longrightarrow$ $\sum b_n$ div.

Example Does $\sum \frac{n}{n^2+3}$ converge / diverge?

Sol^n: Idea $\frac{n}{n^2+3}$ behaves like $1/n$ as $n \to \infty$.

So series should diverge

Claim $\sum \frac{n}{n^2+3}$ diverges

Pf: If $n \ge 2$, $n^2 + 3 \le 2n^2$.

So $\frac{n}{n^2+3} \ge \frac{n}{2n^2} = \frac{1}{2n}$.

$\sum \frac{1}{2n}$ div. So $\sum \frac{n}{n^2+3}$ div.

<u>Th^m 2.26</u> (Limit Comparison test). Let $a_n, b_n > 0$ for $n > m$. Let

$$\alpha = \lim_{n\to\infty} \frac{a_n}{b_n}.$$

Sps. $0 < \alpha < \infty$. Then $\sum a_n$ conv. $\Longleftrightarrow$ $\sum b_n$ conv.

<u>Pf</u>. $\exists$ N. s.t $\forall$ $n > N$.

$$\frac{\alpha}{2} < \frac{a_n}{b_n} < \frac{3}{2}\alpha$$ (Apply limit def^n with $\varepsilon = \frac{\alpha}{2} > 0$).

i.e $n > N \Longrightarrow \begin{cases} a_n > \frac{\alpha}{2} b_n & (*) \\ b_n > \frac{2}{3\alpha} a_n & (**) \end{cases}$

If $\sum a_n$ conv. $\xRightarrow{(*)}$ $\frac{\alpha}{2}\sum b_n$ or $\sum b_n$ conv.

$\sum b_n$ conv $\xRightarrow{(**)}$ $\frac{2}{3\alpha}\sum a_n$ or $\sum a_n$ conv.

<u>Example</u>. Consider $\displaystyle\sum \frac{n^3+1}{4n^5-6n^2+n+1}$.

Let $a_n = \dfrac{n^3+1}{4n^5-6n^2+n+1}$, $b_n = \dfrac{1}{n^2}$.

Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2(n^3+1)}{4n^5-6n^2+n+1}$

$$= \frac{1}{4}$$

Since $\sum \frac{1}{n^2}$ conv. $\Longrightarrow \sum \frac{n^3+1}{4n^5-6n^2+n+1}$ conv.

• Alternating Series

Def^n A series is called alternating if it is of the form $\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 \cdots$

Th^m 2.27 Let $\{a_n\}$ be a seq^n s.t

(1) $a_n \downarrow$

(2) $\lim_{n\to\infty} a_n = 0$

Then $\sum (-1)^{n-1} a_n$ converges. Moreover if $S = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$ then $|S - S_n| \le a_n$.

Pf: Consider the even partials

$$S_{2n} = \sum_{k=1}^{2n} (-1)^{n-1} a_n.$$

Then
$$S_{2(n+1)} = \sum_{k=1}^{2n} (-1)^{n-1} a_n + a_{2n+1} - a_{2n+2}$$

$$= S_{2n} + (a_{2n+1} - a_{2n+2}) \geq S_{2n}$$

Since $a_{2n+1} \geq a_{2n+2}$.

So $s_{2n} \uparrow$

Also $s_{2n} = a_1 - \underbrace{(a_2 - a_3)}_{\geq 0} + \cdots - \underbrace{(a_{2n-2} - a_{2n-1})}_{\geq 0}$

$- a_{2n}$ ≥ 0

$\leq a_1$

So $\{s_{2n}\}$ bounded.

MCT $\Rightarrow$ $\{s_{2n}\}$ converges as $n \to \infty$. Let

$$S = \lim_{n\to\infty} s_{2n}$$

<u>Claim</u>: $s_n \to S$.

<u>Pf</u>: $s_{2m} \xrightarrow{m\to\infty} S$. So $\forall\, \varepsilon > 0$, $\exists\, N_1$ s.t

$$m > N_1 \implies |s_{2m} - S| < \varepsilon/2. \quad (*)$$

Now $s_{2m+1} = s_{2m} + a_{2m+1}$

$$\Rightarrow |s_{2m+1} - S| \leq |s_{2m} - S| + |a_{2m+1}|$$

$\lim\limits_{n\to\infty} a_n = 0 \implies \exists\, N_2$ s.t

$$m > N_2, \quad |a_{2m+1}| < \varepsilon/2. \quad (**)$$

Let $N = \max(2N_1, 2N_2)$

Sps $n > N$.

CASE 1 $n = 2m$. then $m > N_1, N_2$. So,

$$(*) \Rightarrow |s_n - s| = |s_{2m} - s| < \frac{\varepsilon}{2} < \varepsilon.$$

CASE 2 $n = 2m+1$, then $m > N_1, N_2$.

$$(*), (**) \Rightarrow |s_n - s| = |s_{2m+1} - s| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So $n > N \Rightarrow |s_n - s| < \varepsilon$. So $\lim_{n\to\infty} s_n = s$.

Moreover,

$$s - s_n = \sum_{k=n+1}^{\infty} (-1)^{k-1} a_k = \begin{cases} a_{n+1} - (a_{n+2} - a_{n+3}) \cdots & n \text{ is even} \\ -(a_{n+1} - a_{n+2}) - \cdots & n \text{ is odd}. \end{cases}$$

If n is even, then $0 \le s - s_n \le a_{n+1}$

n is odd, then $-a_{n+1} \le s - s_n \le 0$

In both cases $|s - s_n| \le a_{n+1} \le a_n$.

Ex Consider $\sum_{n=1}^{\infty} (-1)^n/n = 1 - \frac{1}{2} + \frac{1}{3} - \cdots$

Here $a_n = 1/n$. Then $a_n \downarrow 0$.

Th^m $\Rightarrow \sum_{n=1}^{\infty} (-1)^n/n$ converges.

<u>Fact</u>: $\sum_{n=1}^{\infty} (-1)^{n-1}/n = \ln 2 \approx 0.693\ldots$

To approximate $\ln 2$ to 3 decimals, we need $|s - s_n| < 10^{-4}$.

We need n large enough that $|a_n| < 10^{-4}$.

i.e. $n > 10000$

$\sum \frac{1}{n}$ is called the harmonic series &

$\sum (-1)^n/n$ the alternating harmonic series.

- <u>Absolute vs Conditional conv.</u>

<u>Th^m 2.28</u> Let $\{a_n\}$ be a seq^n. Then if

$\sum |a_n|$ converges $\Longrightarrow$ $\sum a_n$ converges

<u>Pf</u>: $\sum |a_n|$ converges $\Longrightarrow$ $\forall\, \varepsilon > 0$, $\exists\, N$ s.t.

$$n > m > N, \quad \sum_{k=m}^{n} |a_k| < \varepsilon.$$

By Δ-ineq $\left| \sum_{k=m}^{n} a_k \right| \leq \sum_{k=m}^{n} |a_k| < \varepsilon.$

By CC criteria, $\sum a_k$ conv.

<u>Rk</u> Converse not true. eg $a_n = (-1)^{n-1}/n$

Then $\sum a_n$ conv. <u>but</u> $\sum |a_n| = \sum \frac{1}{n}$ diverges

<u>Example</u>: Consider series $\sum \frac{\sin(n)}{n^2}$

Since $\left|\frac{\sin(n)}{n^2}\right| \le \frac{1}{n^2}$.

by comparison test $\sum |\sin(n)|/n^2$ conv.

$Th^m \Longrightarrow \sum \sin(n)/n^2$ converges

<u>Def^n</u>: 1) A series $\sum a_n$ is called <u>absolutely convergent</u> if $\sum |a_n|$ converges.

2) It is called <u>conditionally</u> <u>convergent</u> if $\sum a_n$ converges but $\sum |a_n|$ diverges.

<u>Eg</u>: $\sum (-1)^{n-1}/n$ is conditionally conv. but

$\sum (-1)^{n-1}/n\sqrt{n}$ is absolutely conv.

<u>Th^m</u> 2.29 (Ratio test). Let $\{a_n\}$ be a seq^n, s.t $a_n \neq 0$, and let

$$\alpha := \limsup \left|\frac{a_{n+1}}{a_n}\right|$$

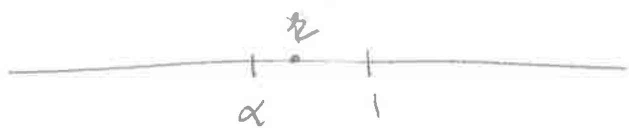
$$\beta = \liminf \left|\frac{a_{n+1}}{a_n}\right|$$

(1) $\sum a_n$ conv. <u>absolutely</u> if $\alpha < 1$.

(2) diverges if $\beta > 1$.

(3). If $\beta \leq 1 \leq \alpha$, test is inconclusive.

Pf: ① Sps $\alpha < 1$. Let $r \in (\alpha, 1)$



Then $\exists N$ (since α is limsup; see Th^m 2.15).

s.t $n > N \Rightarrow \left| \frac{a_{n+1}}{a_n} \right| < r$

$$\text{i.e } |a_{n+1}| < r|a_n|$$
$$< r^2 |a_{n-1}|$$
$$\vdots$$
$$< r^{n-N} |a_{N+1}|.$$

Or for $n \geq N+1$, $|a_n| < r^{n-N-1} |a_{N+1}|$.

Now, since $r < 1$, $\sum_{n=1}^{\infty} r^{n-N-1} |a_{N+1}|$

$$= r^{-N-1} |a_{N+1}| \sum_{n=1}^{\infty} r^n \text{ conv.}$$

geometric series

Comparison test $\Longrightarrow \sum |a_n|$ conv.

(2) If $\beta > 1$, then

$\exists \; N$ s.t $n > N \Rightarrow \left| \dfrac{a_{n+1}}{a_n} \right| > 1$.

$\Rightarrow \forall n > N, \quad |a_n| > |a_{N+1}| \neq 0$.

$\Rightarrow \lim\limits_{n\to\infty} a_n \neq 0$.

So $\sum a_n$ diverges.

(3) Consider $\sum \frac{1}{n}$, $\sum \frac{1}{n^2}$, $a_n = 1/n$, $b_n = 1/n^2$.

In both cases $\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \lim\limits_{n\to\infty} \left| \dfrac{b_{n+1}}{b_n} \right| = 1$.

But $\sum \frac{1}{n}$ diverges while $\sum \frac{1}{n^2}$ conv.

<u>Example</u> $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$

$a_n = 1/n!$ Then $\left| \dfrac{a_{n+1}}{a_n} \right| = \dfrac{n!}{(n+1)!} = \dfrac{1}{n+1} \to 0$.

So $\limsup \left| \dfrac{a_{n+1}}{a_n} \right| = 0 < 1$.

$\Rightarrow \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$ converges.

Def^n: We define the number e as

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

Th^m 2.30 (Root test). Let $\{a_n\}$ be a seq^n and let $\alpha = \limsup_{n\to\infty} |a_n|^{1/n}$.

(1) $\sum a_n$ converges absolutely if $\alpha < 1$.

(2) $\sum a_n$ diverges if $\alpha > 1$.

(3) Inconclusive if $\alpha = 1$.

Pf: Again compare to geometric series

① $\alpha < 1$, Let $r \in (\alpha, 1)$. $\exists$ N s.t

$$n > N \Rightarrow |a_n|^{1/n} < r.$$

$$\Rightarrow |a_n| < r^n \quad \forall n > N.$$

$r < 1 \Rightarrow \sum r^n$ conv.

Comparison test $\Rightarrow \sum |a_n|$ conv.

② If $\alpha > 1$, then $\forall$ N $\exists$ $n > N$ s.t

$$|a_n|^{1/n} > 1 \Leftrightarrow |a_n| > 1.$$

Then $\lim_{n\to\infty} a_n \neq 0$. So $\sum a_n$ diverges

③ Again let $a_n = 1/n$, $b_n = 1/n^2$.

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} |b_n|^{1/n} = 1$$

So, inconclusive since $\sum \frac{1}{n}$ div. & $\sum \frac{1}{n^2}$ conv.

Example: Consider $\sum \frac{n^2}{3^n}$.

Take $a_n = n^2/3^n$. Then

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \frac{n^{2/n}}{3} = \frac{1}{3} < 1.$$

So $\sum \frac{n^2}{3^n}$ converges.

Rk 1) Generally Ratio test works well with factorial & root test with powers.

2). The root test is strictly stronger than ratio test. One can prove that

$$\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < \liminf_{n\to\infty} |a_n|^{1/n} \leq \limsup_{n\to\infty} |a_n|^{1/n}$$
$$\leq \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

If $\limsup_{n\to\infty} |a_{n+1}/a_n| < 1$, then $\limsup_{n\to\infty} |a_n|^{1/n} < 1$.

i.e. ratio works then root test will also work. But instances where root test works but ratio does not.

(see HW 2).

More about e.

Th^m 2.31 $\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$.

Pf: Let $\displaystyle s_n = \sum_{k=0}^{n} \frac{1}{k!}$, $\displaystyle t_n = \left(1+\frac{1}{n}\right)^n$.

Binomial Th^m $\Longrightarrow$

$$\left(1+\frac{1}{n}\right)^n = 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{2!}\cdot\frac{1}{n^2} + \cdots + \frac{n(n-1)\cdots(n-k+1)}{k!}\cdot\frac{1}{n^k} + \cdots + \frac{1}{n^n}$$

$$= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$$

$$\leq s_n.$$

$\Longrightarrow \displaystyle\limsup_{n\to\infty} \leq e$.

Next, if $n \geq m$

$$t_n \geq 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots + \frac{1}{m!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{m-1}{n}\right).$$

As $n \to \infty$

$$\liminf_{n\to\infty} t_n \geq 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{m!}$$

$$= S_m .$$

Now let $m \to \infty$. So $\liminf_{n\to\infty} t_n \geq e$.

So

$$e \leq \liminf_{n\to\infty} t_n \leq \limsup_{n\to\infty} t_n \leq e .$$

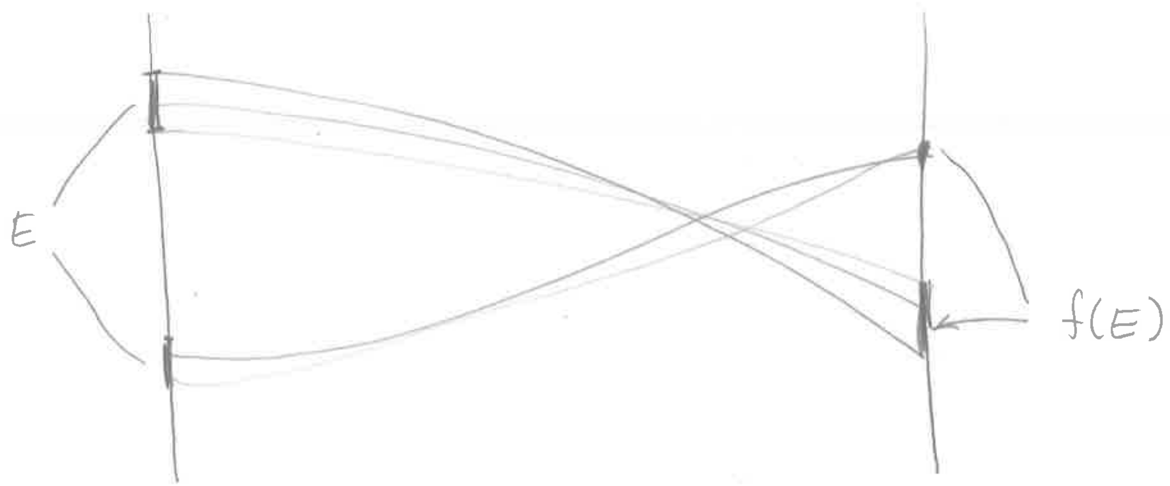
$$\Longrightarrow \quad \liminf_{n\to\infty} t_n = \limsup_{n\to\infty} t_n = e .$$

So $\lim_{n\to\infty} t_n = e$.

Chapter - 3 Limits and Continuity

· Functions on $\mathbb{R}$.

Def^n. Let $E \subset \mathbb{R}$. A (real valued) function on E, denoted by $f : E \to \mathbb{R}$, is a rule that assigns a unique real number $y = f(x)$, for every $x \in E$.



We call E the domain of f and

$$f(E) := \{ f(x) \mid x \in E \}$$

the range. For any $S \subset E$, we call

$$f(S) := \{ f(x) \mid x \in S \}$$

the image of S under f, and for any $S \subset \mathbb{R}$ we call

We say f is cont. on E iff f is cont at $p \in E$ $\forall$ $p \in E$.

Examples 1) Consider $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = ax + b$.

Claim: f is cont. on $\mathbb{R}$.

Aim: Given $\varepsilon > 0$ and $p \in \mathbb{R}$, find $\delta > 0$ s.t. $|x - p| < \delta \implies |f(x) - f(p)| < \varepsilon$.

Now, $|f(x) - f(p)| = |a(x - p)| = |a||x - p|$.

If $|x - a| < \delta$, then $|a||x - p| < \delta|a|$.

If $a \neq 0$, then let $\delta = \varepsilon/|a|$.

If $a = 0$, then $|f(x) - f(p)| = 0 < \varepsilon$ for any $\varepsilon > 0$, so δ could be taken to be anything

In either case

$|x - p| < \delta \implies |f(x) - f(p)| < \varepsilon$.

So f cont at p.

Note: In this example δ was independent of p.

2) Now, let $f(x) = x^2 : \mathbb{R} \longrightarrow \mathbb{R}$.

Claim 1 f is cont. at $p = 2$.

$$f^{-1}(S) = \{ x \in E \mid f(x) \in S \}.$$

the inverse image of S under f.

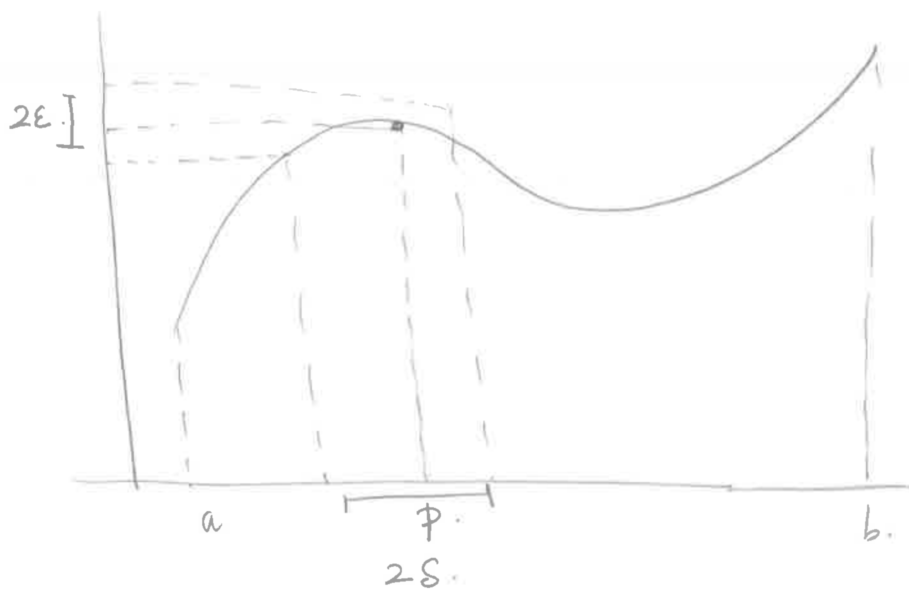
We say f is 1-1 or <u>injective</u> if

$$f(x) = f(y) \iff x = y.$$

We say f is onto or <u>surjective</u> if $f(E) = \mathbb{R}$.

- <u>Continuous functions</u>

Intuitively f is cont. if small perturbations in 'x' cause only small perturbations in $f(x)$.



<u>Def^n</u>: Let $f: E \longrightarrow \mathbb{R}$, and $p \in E$. We say f is <u>continuous</u> at p if $\forall\, \varepsilon > 0$, $\exists\ \delta := \delta(p, \varepsilon) > 0$

s.t

$$\left.\begin{array}{l} |x - p| < \delta \\ x \in E \end{array}\right\} \Longrightarrow |f(x) - f(p)| < \varepsilon.$$

Else, we say f is discontinuous at $p \in E$.

Aim: Given $\varepsilon > 0$, need to find δ s.t.

$$|x-2| < \delta \implies |x^2 - 4| < \varepsilon.$$

Note, $|x^2 - 4| = |x-2||x+2|$.

So if $|x-2| < \delta$, then $|x^2-4| < \delta|x+2|$.

By Δ-ineq, $|x+2| = |x-2+4| < |x-2|+4$
$< 4+\delta$.

So if $|x-2|<\delta$, then $|x^2-4| < \delta(4+\delta)$.

Sps $\delta < 1$, then $\delta(4+\delta) < 5\delta$.

If this has to be smaller than ε, we need $\delta < \varepsilon/5$. So we take any

$$\delta < \min(1, \varepsilon/5).$$

Then $|x-2| < \delta \implies |x^2-4| < \varepsilon$.

Claim 2 f is cont on $\mathbb{R}$.

Pf: Let $\varepsilon > 0$ and $p \in \mathbb{R}$. Then

$|x^2 - p^2| = |x-p||x+p|$.

If $|x-p| < \delta$, then $|x^2-p^2| < |x+p|\delta$.

Also, by Δ-ineq

$$|x-p| < \delta \implies |x+p| = |x-p+2p|$$
$$\leq |x-p| + 2|p|$$
$$< \delta + 2|p|.$$

So $|x - p| < \delta \implies |x^2 - p^2| < (2|p| + \delta)\delta$.

If $\delta < 1$, then $(2|p| + \delta)\delta < (2|p| + 1)\delta$.

which we want to be smaller than ε.

So let

$$\delta < \min\left(1, \frac{\varepsilon}{2|p|+1}\right).$$

Then $|x - p| < \delta \implies |x^2 - p^2| < \varepsilon$. & f is cont. at p.

Rk: The δ in this example depends on both ε & p. If p is large, δ becomes smaller.

• Basic Properties of cont. functions

Thm 2.32. Let $f: E \to \mathbb{R}$. Then f is cont. at $p \iff$ for any seqn $\{x_n\}$ in $E \setminus \{p\}$ $x_n \to p$, we have

$$\lim_{n\to\infty} f(x_n) = f(p).$$

Pf $\Rightarrow$: Sps f is cont. at p, and let $x_n \in E$ s.t. $x_n \to p$. Let $\varepsilon > 0$. f cont $\implies \exists\ \delta$ s.t

$$\left.\begin{array}{l} |x - p| < \delta \\ x \in E \end{array}\right\} \implies |f(x) - f(p)| < \varepsilon. \quad (*)$$

$x_n \longrightarrow p \implies \exists \quad N = N(\delta)$ s.t.

$n > N \implies |x_n - p| < \delta.$ $\quad (**)$

So $(*)$ and $(**) \implies$

$n > N \implies |f(x_n) - f(p)| < \varepsilon.$

& hence $\lim_{n\to\infty} f(x_n) = f(p).$

$\underline{\Longleftarrow}$ Sps. f is NOT cont. at p. Then $\exists\ \varepsilon > 0$ s.t for any $\delta > 0$; $\exists\ x_\delta \in E$ s.t

$|x_\delta - p| < \delta$, but $|f(x_\delta) - f(p)| > \varepsilon.$

Clearly $x_\delta \in E\setminus\{p\}$. Applying this with $\delta = 1/n$, $\forall\ n \in \mathbb{N}$, $\exists\ x_n \in E\setminus\{p\}$ s.t

$|x_n - p| < 1/n$ but $|f(x_n) - f(p)| > \varepsilon.$

Clearly $x_n \longrightarrow p$, but $\lim_{n\to\infty} f(x_n) \neq f(p).$

Contradiction!

<u>Rk</u> A point $p \in E$ is called <u>isolated</u> if there is no seq^n $\{x_n\}$ in $E\setminus\{p\}$ s.t $x_n \longrightarrow p$.

Any $f : E \longrightarrow \mathbb{R}$ is automatically cont. at $p \in E$ if p is isolated. Since if p is isolated $\exists\ \delta > 0$ s.t $(p-\delta, p+\delta) \cap E = \{p\}.$

Thm 2.33: Let $f, g : E \to \mathbb{R}$ be cont. functions at $p \in E$, and let $a \in \mathbb{R}$. Then

1. $a \cdot f$ and $|f|$ are cont. at p.
2. $f + g$ is cont at p.
3. $f \cdot g$ is cont at p.
4. f/g is cont at p if $g(p) \neq 0$.

Pf: These follow from Thm 2.32 and algebraic properties of sequences.

Cor 2.34: 1. The absolute value function

$$|x| := \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

is cont. on $\mathbb{R}$.

2. Polynomial functions $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ are cont. on $\mathbb{R}$.

3. Rational functions $R(x) = p(x)/q(x)$ where p & q are polynomials are cont. wherever $q(x_0) \neq 0$

Pf: 1. Follows from x being a cont. function and Thm 2.33 (1).

2) Since $f(x) = x$ is cont. Th^m 2.33 (3) applied 'n' times $\Longrightarrow$ x^n is cont. Then Th^m 2.33((1) + (2)) $\Longrightarrow$ $p(x)$ is cont.

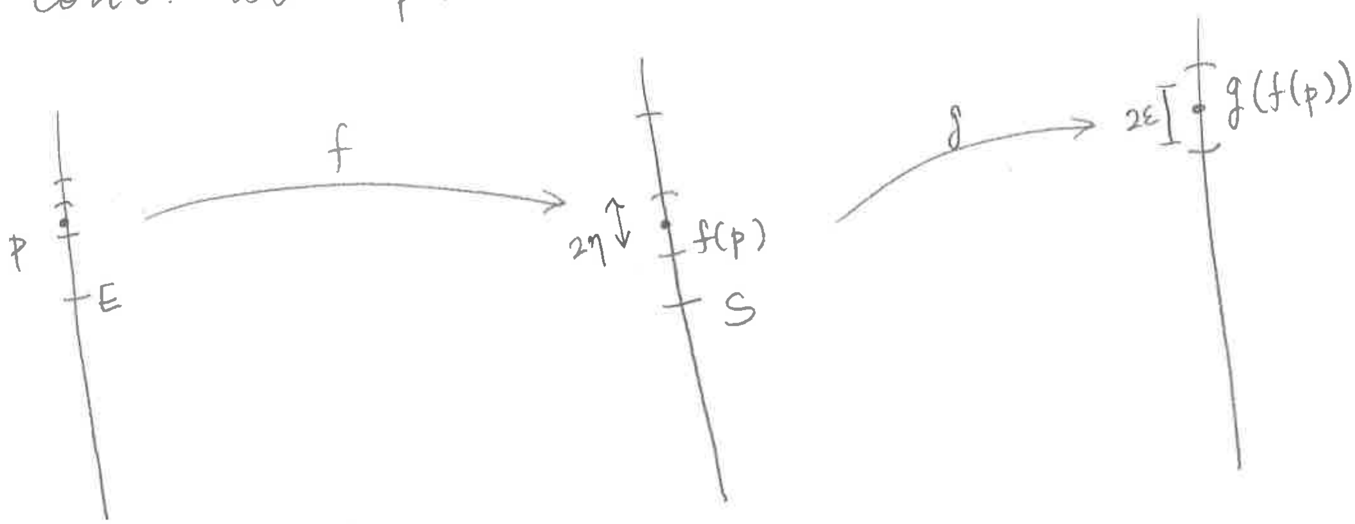
3) Follows from (2) above & Th^m 2.33 (4).

<u>Def^n</u> Let $E \subset \mathbb{R}$, $f : E \to \mathbb{R}$ and $f(E) \subset S$. If $g : S \to \mathbb{R}$, we define the <u>composite function</u> $g \circ f : E \to \mathbb{R}$ by

$$g \circ f(x) = g(f(x)).$$

<u>Th^m 2.35</u> Let f and g as above. If f is cont. at $p \in E$ and g cont. at $f(p) \in S$, then $g \circ f$ is cont. at p.

<u>Pf</u>:



Let $\varepsilon > 0$. Since g is cont. at $f(p)$, $\exists\ \eta > 0$ s.t

$$\begin{array}{l}|y - f(p)| < \eta \\ y \in S\end{array} \Longrightarrow |g(y) - g \circ f(p)| < \varepsilon. \quad (*)$$

Since f is cont at p, $\exists\ \delta > 0$ s.t.

$$\left.\begin{array}{l} |x-p| < \delta \\ x \in E \end{array}\right\} \Longrightarrow |f(x) - f(p)| < \eta \quad (**)$$

Now if $\left.\begin{array}{l} |x-p| < \delta \\ x \in E \end{array}\right\} \Rightarrow |f(x) - f(p)| < \eta$.

Since $f(x) \in S \overset{(*)}{\Longrightarrow} |g \circ f(x) - g \circ f(p)| < \varepsilon$.

So $g \circ f$ is cont. at p.

• <u>Trigonometric functions and exponentials.</u>

We'll formally define $\sin(x), \cos(x) : \mathbb{R} \to [-1,1]$ later.

<u>FACT</u>: $\sin(x)$ and $\cos(x)$ are continuous periodic functions on $\mathbb{R}$.

The period is defined to be 2π. i.e

$$\left.\begin{array}{l} \sin(x + 2\pi) = \sin(x) \\ \cos(x + 2\pi) = \cos(x) \end{array}\right. \quad \forall\ x \in \mathbb{R}.$$

Similarly we will define e^x later.

<u>FACT</u>: e^x is cont.

From now on we'll use these functions and all properties learnt in high-school even though we have not formally defined them.

Combining with Th^m 2.35 & Th^m 2.33 we can construct many more cont. functions.

e.g: $e^{\sin x}$, $\tan x = \sin x / \cos x$, $x \neq n\pi/2$,

$\sin(\cos x)$ ....