WEEK - 1

Chapter - 1 Number systems.

- **Natural numbers & rationals.**

We denote the set of natural numbers by

$$\mathbb{N} = \{1, 2, 3, \dots\},$$

and the set of integers by

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, \dots\}.$$

We'll denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \dots\}$.
The natural numbers can be axiomatically defined using the Peano axioms. One consequence is the foll.

- Axiom of induction: If $S \subseteq \mathbb{N}$ with the property

(1) $1 \in S$
(2) If $n \in S$, then $n+1 \in S$.

Then $S = \mathbb{N}$.

- Principle of mathematical induction: Consider statements P_n indexed by $n \in \mathbb{N}$ s.t.

(1) (Base Case) P_1 is true.

(2) (Induction Step) If P_n is true, then P_{n+1} is true.

Then P_n is true for all $n \in \mathbb{N}$.

Example: $1+2+3+\cdots+n = \frac{n(n+1)}{2}$.

Pf: Let P_n be the above statement.

(1) P_1: $1 = 1$ Obviously true.

(2) Sps P_n is true. Then

$$1+2+3+\cdots n+1 = 1+2+\cdots+n+n+1$$

$$\overset{P_n}{=} \frac{n(n+1)}{2} + n+1.$$

$$= \frac{(n+1)(n+2)}{2} = \frac{(n+1)((n+1)+1)}{2}.$$

So P_{n+1} is true.

Induction $\Longrightarrow$ P_n is true for all n.

Rk: We'll assume that we know how to add, subtract & multiply two integers to obtain other integers.

Notation: $\alpha \in A$: α is an element of A.

$\forall$: for all

$\exists$: there exists.

<u>Def^n</u>: We define $\mathbb{Q}$, the set of <u>rational numbers</u> by
$$\mathbb{Q} := \left\{ \frac{p}{q} \;\middle|\; p \in \mathbb{Z},\ q \in \mathbb{N} \right\}.$$
s.t
$$\frac{p}{q} = \frac{r}{s} \iff ps = qr.$$

<u>Rk</u>: If $p/q \in \mathbb{Q}$ and $n \in \mathbb{Z} \setminus \{0\}$, then $p/q = np/nq$. So we generally assume that p & q have no common factors i.e $\gcd(p, q) = 1$.
Also, $\mathbb{Z} \subseteq \mathbb{Q}$, by taking $q = 1$.

<u>Def^n</u> (For p/q, $r/s \in \mathbb{Q}$ define

(1) <u>Addition</u> $\quad \dfrac{p}{q} + \dfrac{r}{s} = \dfrac{ps + qr}{qs}$.

(2) <u>Multiplication</u> : $\quad \dfrac{p}{q} \cdot \dfrac{r}{s} = \dfrac{pr}{qs}$.

(3) <u>Order</u> : $\quad p/q \leq r/s \iff ps \leq rq$.

If $p/q \neq r/s$ we write $p/q < r/s$.

These satisfy foll. properties.

<u>F1</u>: (Closure) $\alpha, \beta \in \mathbb{Q} \implies \alpha + \beta,\ \alpha \cdot \beta \in \mathbb{Q}$

<u>F2</u> (Commutativity): $\alpha, \beta \in \mathbb{Q} \implies$
$$\alpha + \beta = \beta + \alpha$$
$$\alpha \cdot \beta = \beta \cdot \alpha.$$

F3: (Associativity) $\alpha, \beta, \gamma \in \mathbb{Q} \Rightarrow \alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$

$$\alpha \cdot (\beta \gamma) = (\alpha \beta) \cdot \gamma$$

F4 (Identity element) $\alpha + 0 = \alpha$, $\alpha \cdot 1 = \alpha$ for all $\alpha \in \mathbb{Q}$

F5 (Inverses). $\forall \alpha \in \mathbb{Q}$, $\exists -\alpha \in \mathbb{Q}$ s.t

$\alpha + (-\alpha) = 0$. (Infact $-\alpha = -1 \cdot \alpha$)

$\forall \alpha \in \mathbb{Q} \setminus \{0\}$ $\exists$ $\alpha^{-1} \in \mathbb{Q}$ s.t.

$\alpha \cdot \alpha^{-1} = 1$ (Infact if $\alpha = p/q$, $\alpha^{-1} = q/p$)

F6 (Distributive) $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$

Any set S with binary operations $+$, $\cdot$ satisfying F1 - F6 is called a <u>field</u>.

$(\mathbb{Q}, \leq)$ has the foll. properties

O1 $\forall \alpha, \beta \in \mathbb{Q}$ either $\alpha < \beta$, $\alpha = \beta$ or $\beta < \alpha$.

O2 (Transitivity). $\alpha < \beta$, $\beta < \gamma \Rightarrow \alpha < \gamma$

O3 $\alpha \leq \beta \Longrightarrow \alpha + \gamma \leq \beta + \gamma$

O4 $\alpha \leq \beta$, $\gamma \geq 0 \Longrightarrow \alpha \gamma \leq \beta \gamma$.

Any set S with an order satisfying O1 - O4 is called an <u>ordered set</u>.

<u>Def^n</u>: A set F together with $(+, \cdot, \leq)$ is called an <u>ordered</u> <u>field</u> if it satisfies F1 - F6 & O1 - O4.

<u>E.g</u> $\mathbb{Q}$ is an ordered field

$\mathbb{Z}$ is not a field but is an ordered set.

• <u>Inadequacies of $\mathbb{Q}$</u>.

1) Irrationality of $\sqrt{2}$.

<u>Th^m</u> There is no $\alpha \in \mathbb{Q}$ s.t $\alpha^2 = 2$.

<u>Pf</u>: (Proof by contradiction). Sps there is such an $\alpha \in \mathbb{Q}$. Let $\alpha = p/q$, $\gcd(p, q) = 1$.

$\alpha^2 = 2 \implies p^2 = 2q^2$. $(*)$

$\implies 2$ divides p^2 (written as $2 \mid p^2$).

$\underset{\text{prime}}{2 \text{ is}} \implies 2 \mid p$.

Let $p = 2m$. Then $(*) \implies 4m^2 = 2q^2$

$\implies q^2 = 2m^2$.

Again same argument $\implies 2 \mid q$.

So $2 | p$ & $2 | q$. But $\gcd(p, q) = 1$

Contradiction!

<u>Moral</u>: Certain algebraic equations involving rationals have <u>no</u> rational solutions.

2) <u>Incompleteness</u>.

<u>Def^n</u> (Upper / lower bound). Let S be an ordered set, and let $A \subseteq S$. An element $\beta \in S$ is an upper bound (u.b) (resp. lower bound) for A if $\alpha \leq \beta$ $\forall \alpha \in A$ (resp. $\alpha \geq \beta$).
We then say that A is bounded above (resp. below)

<u>Example</u>: $S = \mathbb{Q}$, $A = \{\alpha \in \mathbb{Q} \mid \alpha^2 < 2\}$.
Then $3/2$, 2 etc are upper bounds.

<u>Def^n</u>: (Supremum / Infimum): Sps S is an ordered set & $A \subseteq S$ is <u>bounded above</u>. Sps $\exists \beta \in S$ s.t

(1) β is an u.b for A.

(2) If γ is another u.b, then $\gamma \geq \beta$.

Then β is called the least upper bound (l.u.b) or <u>supremum</u> (sup) of A.

We write $\beta = \sup A$. If $\beta \in A$ we also write $\boxed{\beta = \max A}$

Similarly, if A is bounded below, we define the <u>greatest</u> <u>lower</u> <u>bound</u> (g.l.b) or <u>infimum</u> ($\inf A$)., and if $\beta = \inf A \in A$, we also say $\boxed{\beta = \min A}$

<u>Rk</u>: ① A sup/inf need not belong to A.

<u>e.g</u>: $S = \mathbb{Q}$, $A = \{\alpha \in \mathbb{Q} \mid 0 \leq \alpha < 1\}$.

Then $\sup A = 1 \notin A$.

② A sup/inf <u>need not exist</u>.

<u>e.g</u>: $S = \mathbb{Q}$, $A = \{\alpha \in \mathbb{Q} \mid \alpha^2 < 2\}$.

Then A is bounded above.

<u>Claim</u>: A does not have a sup in $\mathbb{Q}$.

<u>Pf</u>: Sps it does have a sup. i.e $\exists\, \beta \in \mathbb{Q}$ s.t $\beta = \sup A$.

<u>Step 1</u>: If $\beta^2 < 2$, then $\alpha = \beta + \frac{2-\beta^2}{6} \in \mathbb{Q}$ satisfies $\alpha^2 < 2$. So $\alpha \in A$, but $\alpha > \beta$. So β is <u>not</u> u.b for A.

<u>Step 2</u> If $\beta^2 > 2$, then $\alpha = \beta - \frac{\beta^2 - 2}{6} \in \mathbb{Q}$ s.t $\alpha^2 > 2$. but $\alpha < \beta$, and is an u.b for A. So β is not the least upper bound.

<u>Step 3</u>: So $\beta^2 = 2$. Contradiction since $\beta \in \mathbb{Q}$.

This proves claim.

<u>Def^n</u>: An ordered set S is said to have the <u>least upper bound property</u> or <u>completeness</u> property if <u>every</u> upper bounded set has a supremum in S.

<u>Th^m</u>: There is a <u>unique</u> <u>ordered</u> <u>field</u> $(\mathbb{R}, +, \cdot, \leq)$ has (1) the completeness property

(2) $\mathbb{Q} \subseteq \mathbb{R}$ is an ordered subfield i.e. $+, \cdot, \leq$ restricted to $\mathbb{Q}$ are the usual addition, multiplication & order on $\mathbb{Q}$.

<u>Def^n</u> The unique field above is called the field of <u>real</u> <u>numbers</u>.

- <u>Basic Properties of $\mathbb{R}$ as an ordered field</u>

<u>Th^m</u> 1.1 Let $a, b, c \in \mathbb{R}$

① $a + c = b + c \iff a = b$.

② $a \cdot 0 = 0$

③ $(-a) b = -ab$.

4. $(-a)(-b) = ab$.
5. $ac = bc,\ c \neq 0 \implies a = b$.
6. $ab = 0 \implies a = 0$ or $b = 0$.
7. $a \leq b \implies -b \leq -a$.
8. $a \leq b,\ c \leq 0 \implies bc \leq ac$.
9. $0 \leq a,\ 0 \leq b \implies 0 \leq ab$.
10. $a^2 \geq 0$ and $a^2 = 0 \iff a = 0$.
11. $0 < 1$.
12. $0 < a$, then $a^{-1} > 0$
13. $0 < a < b \implies 0 < b^{-1} < a^{-1}$.

Pf: Exercise.

For instance to prove (10).

CASE 1: If $a \geq 0$. Then $O4 \implies a^2 \geq 0 \cdot a = 0$.
$a = a$.

CASE 2: If $a < 0$, Then (7) $\implies -a > 0$.

$O4$ applied to $-a = -a \implies (-a)(-a) > 0$
$\implies a^2 > 0$.

Def^n (Absolute value). Define

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases}$$

Th^m 1.2 ① $|a| \geq 0 \quad \forall\, a \in \mathbb{R}$.

$|a| = 0 \iff a = 0$

② $|-a| = |a|$.

③ $|ab| = |a||b| \quad \forall\, a, b \in \mathbb{R}$

④ (Triangle inequality). $|a+b| \leq |a| + |b|$.

⑤ (Reverse triangle ineq) $|b-a| \geq \big||b| - |a|\big|$.

Pf (1), (2), (3) Exercise.

(4). Note $\quad -|a| \leq a \leq |a|$.

$\quad -|b| \leq b \leq |b|$.

$\Rightarrow \quad -(|a|+|b|) \leq a+b \leq |a|+|b|$.

If $a+b \geq 0$, Then $a+b = |a+b|$, so $|a+b| \leq |a| + |b|$

If $a+b < 0$ Then $a+b = -|a+b|$ So

$-(|a|+|b|) \leq -|a+b|$, or $|a+b| \leq |a|+|b|$

<u>Def^n</u> (Distance). Given $a, b \in \mathbb{R}$ we define the distance between them as

$$d(a,b) = |b - a|.$$

<u>Properties</u>: D1 (Symmetry) $d(a,b) = d(b,a)$.

D2 (Positivity) $d(a,b) > 0$ if $a \neq b$
$d(a,a) = 0$.

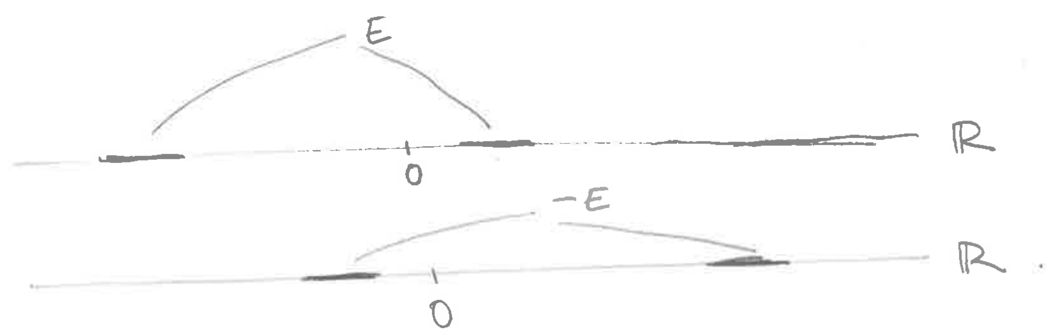
D3 (Triangle ineq). $d(a,b) \leq d(a,c) + d(b,c)$.

• <u>Consequences of the completeness axiom</u>.

① <u>Existence of infimum</u>

<u>Th^m</u> 1.3 Let $E \subset \mathbb{R}$ be a set bounded below. Then $\inf E$ exists in $\mathbb{R}$.

<u>Pf</u>: Define $-E = \{-x \mid x \in E\}$.



Claim 1 $-E$ is bounded above.

Pf: Since E is bounded below, $\exists\ \alpha \in \mathbb{R}$ s.t

$$x \geq \alpha \quad \forall\ x \in E.$$

$$\Rightarrow \quad -x \leq -\alpha \quad \forall\ x \in E$$

$$\Rightarrow \quad y \leq -\alpha \quad \forall\ y \in -E$$

So $-\alpha$ is an u.b for $-E$.

Completeness $\Longrightarrow$ $\exists\ \beta$ s.t $\sup(-E) = \beta$.

Claim: $\inf E = -\beta$.

Pf: By above argument $-\beta$ is lower bound for E. Suppose it is not inf. Then $\exists\ \gamma$ lower bound s.t

(1) $x \geq \gamma \quad \forall\ x \in E$

(2) $\gamma > \beta$.

But then (1) $y \leq -\gamma \quad \forall\ y \in -E$

(2) $-\gamma < -\beta$

So $-\beta$ cannot be l.u.b for $-E$. Contradiction

Example: $E = \{x \in \mathbb{Q} \mid x^2 < 2\}$.

Clearly $x \leq 2 \quad \forall\ x \in E$

$x \geq -2 \quad \forall\ x \in E$

So E is bounded above and below. So $\sup E$ and $\inf E$ exist.

Claim: If $\beta = \sup E$, then $\beta^2 = 2$
$+\beta = \inf E$

Pf: Step 1. If $\beta^2 < 2$, then $\alpha = \beta - \frac{2-\beta^2}{6} \in \mathbb{Q}$ satisfies $\alpha^2 < 2$, $\alpha > \beta$. Since $\alpha \in E$, β could not have been an u.b. So Contradiction

Step 2. If $\beta^2 > 2$, then $\alpha = \beta - \frac{\beta^2 - 2}{6} \in \mathbb{R}$ satisfies $\alpha < \beta$ but α is an u.b for E. So β could not be the l.u.b. Contradiction.

Hence $\beta^2 = 2$. By the argument in the proof, we see $\inf E = -\beta$.

2) Archimedean Property

Thm 1.4 If $a, b \in \mathbb{R}$ with $a, b > 0$. Then $\exists\ n \in \mathbb{N}$ s.t

$$na > b.$$

Pf: $\boxed{\text{Sps not}}$. Then $\exists\ a > 0$, $b > 0$ s.t. $na \leq b$ $\forall\ n \in \mathbb{N}$. Let

$$S = \{na \mid n \in \mathbb{N}\}$$

Then b is an u.b for S. Let $\beta = \sup S$ which exists by completeness. Consider $\beta - a$. Since β is l.u.b, $\overset{a>0}{\exists}$ $s \in S$ s.t $s > \beta - a$. (else $\beta - a$ would be an u.b of S smaller than β). So $\exists$ $m \in \mathbb{N}$ s.t

$$\beta - a < ma$$

or

$$\beta < (m+1)a$$

But $(m+1)a \in S$, So β cannot be an u.b. Contradiction!

<u>Cor</u> 1.5 ① If $a > 0$, then $\exists$ $n \in \mathbb{N}$ s.t $\frac{1}{n} < a$.

② If $b > 0$, then $\exists$ $n \in \mathbb{N}$ s.t $b < n$.

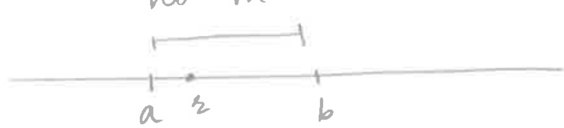
<u>Pf</u>: ① Apply Th^m with $b = 1$.

② Apply Th^m with $a = 1$.

③ <u>Density of $\mathbb{Q}$</u>

<u>Th^m 1.5</u> If $a < b$, $a, b \in \mathbb{R}$. Then $\exists$ $r \in \mathbb{Q}$ s.t $a < r < b$.

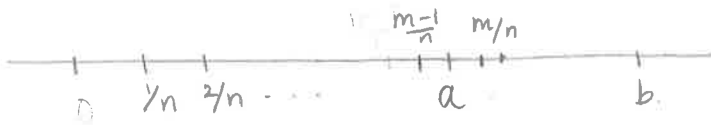
no matter how small $|b-a|$ is



Aim: Find $m \in \mathbb{Z}$, $n \in \mathbb{N}$ s.t. $a < \frac{m}{n} < b$

or $na < m < nb$.

Idea: Choose n large enough so $1/n$ is small



enough to "step over" a but not cross b in consecutive steps.

Pf: Since $b - a > 0$, by Cor 1.5, $\exists\, n \in \mathbb{N}$ s.t

$$\frac{1}{n} < b - a. \qquad (*)$$

Let $m \in \mathbb{N}$ s.t

$$(m-1) \leq na < m. \qquad (**)$$

Claim $b > m/n$.

Pf: $(*) \Rightarrow b > a + \frac{1}{n}$.

$(**) \Rightarrow a \geq \frac{m}{n} - \frac{1}{n}$.

So $b > \frac{m}{n} - \frac{1}{n} + \frac{1}{n} = \frac{m}{n}$.

Def^n A real number α is called <u>irrational</u> if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. We denote

$$\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$$

<u>Cor 1.6</u> Given $a, b \in \mathbb{R}$, $a < b$. $\exists$ $\alpha \in \mathbb{I}$ s.t

$a < \alpha < b$.

<u>Pf</u>: Denote $\sqrt{2} = \sup\{x \in \mathbb{R} \mid x^2 < 2\}$. Then

$a - \sqrt{2} < b - \sqrt{2}$.

By Th^m 1.5, $\exists$ $r \in \mathbb{Q}$ s.t

$$a - \sqrt{2} < r < b - \sqrt{2}.$$

Let $\alpha = r + \sqrt{2}$, Then $a < \alpha < b$.

<u>Claim</u>: $\alpha \in \mathbb{I}$

<u>Pf</u>: If not, then $\alpha \in \mathbb{Q}$. But then $\sqrt{2} = \alpha - r$ has to be a rational since $\mathbb{Q}$ is a <u>field</u> & addition of two rationals (α and $(-r)$) gives another rational. Contradiction!

4) <u>Existence of n^{th} roots</u>

<u>Th^m 1.7</u> Given any $\alpha \in \mathbb{R}$, $\alpha > 0$, and any $n \in \mathbb{N}$, $\exists$ $\beta \in \mathbb{R}$ s.t $\beta^n = \alpha$.

<u>Pf</u>: Let $E = \{x \in \mathbb{R} \mid x^n < \alpha\}$.

<u>Claim 1</u> E is bounded above.

<u>Pf</u>. If $\alpha < 1$, then E is bounded above by 1 since $x > 1 \Rightarrow x^n > 1 > \alpha$.

If $\alpha > 1$, then E is bounded above by α. Since if $x > \alpha$, then $x^n > \alpha^n > \alpha$.

Now, let

$$\beta = \sup E.$$

<u>Claim 2</u> $\beta^n = \alpha$.

The proof relies on the identity

$$b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + \cdots + ba^{n-2} + a^{n-1})$$
$$\leq (b-a)\, n \cdot b^{n-1} \quad \text{if} \quad 0 < a < b. \quad (*)$$

<u>Pf of Claim 2.</u> If not. Then

<u>CASE 1</u> $\beta^n < \alpha$.

<u>Idea</u>: Choose $\varepsilon > 0$ s.t $(\beta + \varepsilon)^n < \alpha$.

By $(*)$ $$(\beta+\varepsilon)^n \leq \beta^n + n\varepsilon \cdot (\beta+\varepsilon)^{n-1}$$
$$\leq \beta^n + n\varepsilon(\beta+1)^{n-1} \quad \text{if} \quad \varepsilon < 1.$$

Pick $\varepsilon < \dfrac{\alpha - \beta^n}{n(\beta+1)^{n-1}}$. (This uses that $\mathbb{R}$ has no gaps)

Then $(\beta+\varepsilon)^n < \alpha$.

So $\beta + \varepsilon \in E$. But $\beta + \varepsilon > \beta$ contradicting that β is an u.b for E.

CASE 2 $\beta^n > \alpha$.

Idea: Choose $\varepsilon > 0$ s.t $(\beta - \varepsilon)^n > \alpha$. Then $\beta - \varepsilon$ would be an u.b for E smaller than β, contradicting β being l.u.b.

Again by (*)

$$(\beta - \varepsilon)^n \geq \beta^n - \varepsilon n \beta^{n-1}.$$

Pick $\varepsilon < \dfrac{\beta^n - \alpha}{n\beta^{n-1}}$. Then $(\beta - \varepsilon)^n > \alpha$.

Done!

Def^n: Given $\alpha > 0$, its' n^{th} root $(n \in \mathbb{N})$ is a positive real number $\beta > 0$ s.t $\beta^n = \alpha$.

We write $\beta = \alpha^{1/n} = \sqrt[n]{\alpha}$

Cor 1.8 1) $\beta_1, \beta_2 > 0$ s.t $\beta_1^n = \beta_2^n$. Then $\beta_1 = \beta_2$, so n^{th} roots are unique

2) If $a, b > 0$, Then

$$(ab)^{1/n} = a^{1/n} b^{1/n}.$$

- Extended reals. $\mathbb{R}^* = \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$.

with the extended order.

$$-\infty < a < \infty \quad \forall\ a \in \mathbb{R}.$$

Then. ∞ is an u.b for any $E \subset \mathbb{R}$.

$-\infty$ is a l.b for any $E \subset \mathbb{R}$.

We extend the definition of sup & inf s.t

1) $\sup E = \infty$ if $E \subset \mathbb{R}$ is not bounded above

2) $\inf E = -\infty$ if $E \subseteq \mathbb{R}$ is not bounded below.

$\mathbb{R}^*$ does not form a field.

Chapter - 2 Sequences & Series

<u>Def^n</u>: A <u>sequence</u> (seq^n) is a <u>function</u> $f: \mathbb{N} \longrightarrow \mathbb{R}$.
We usually denote it by it's <u>terms</u> $a_n = f(n)$.
and write the sequence as $\{a_n\}_{n=1}^{\infty}$.

<u>Ques</u> What does it mean to say that $\{a_n\}_{n=1}^{\infty}$ converges to a value L?

<u>Examples</u>: 1) $a_n = 1/\sqrt{n}$, $n = 1, 2, \ldots$

First few terms are $(1, 1/\sqrt{2}, 1/\sqrt{3}, 1/\sqrt{4}, \ldots)$.
The terms seem to "approach" 0.

2) $a_n = (-1)^n = \begin{cases} 1, & n \text{ even} \\ -1 & n \text{ odd}. \end{cases}$

Here the terms are $(-1, 1, -1, 1, -1, \ldots)$.
The sequence jumps around, and it appears to <u>not</u> approach a single value.

<u>Rk</u>: It is neccessary to distinguish between a <u>sequence</u> and it's set of values. e.g in the previous case, the seq^n is $(-1, 1, -1, \ldots)$ while set of values is $\{-1, 1\}$.

<u>Def^n</u> A seq^n $\{a_n\}_{n=1}^{\infty}$ is said to <u>converge</u> to L as $n \to \infty$ if $\forall\ \varepsilon > 0$, $\exists\ N = N(\varepsilon) \in \mathbb{N}$

s.t

$$n > N \implies |a_n - L| < \varepsilon.$$

We write $L = \lim_{n\to\infty} a_n$ or $a_n \longrightarrow L$, and call L the limit of $\{a_n\}_{n=1}^{\infty}$ as $n \to \infty$.

If there is no such L we say $\{a_n\}_{n=1}^{\infty}$ diverges.

<u>Rk</u>: If $a_n \to L$ then no matter how small ε is, "eventually" all a_n fall in $(L-\varepsilon, L+\varepsilon)$.

<u>Examples</u> 1) Consider $a_n = 1/\sqrt{n}$. Intuitively $a_n \xrightarrow{n\to\infty} 0$. Let us "prove" this using above definition. To get a feel.

<u>$\varepsilon = 1/10$</u>. We need an N s.t $\forall\ n > N$. $1/\sqrt{n} \in (-1/10, 1/10)$. We can take $N = 100$.

i.e. $\forall\ n > 100$, $\left|\frac{1}{\sqrt{n}} - 0\right| < \frac{1}{10}$.

$\varepsilon = 1/100$ Need N s.t $n > N \Rightarrow \frac{1}{\sqrt{n}} \in \left(\frac{-1}{100}, \frac{1}{100}\right)$

Clearly $N = 10000$ works.

and so on.

Claim: $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

Pf: Let $\varepsilon > 0$ be arbitrary.

Aim: Find N s.t $\forall\, n > N$, $\frac{1}{\sqrt{n}} < \varepsilon$ or $n > \frac{1}{\varepsilon^2}$.

Let N be any natural number $> 1/\varepsilon^2$.

Then.

$$n > N \implies \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}} < \varepsilon.$$

So $n > N \implies \left| \frac{1}{\sqrt{n}} - 0 \right| < \varepsilon$.

$$\implies \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0.$$

2) If $a_n = (-1)^n$, then $\{a_n\}$ is divergent.

If not, then $\lim_{n\to\infty} a_n = L$.

CASE 1 $L = -1$. If $\varepsilon = 1/2$. then infinitely many a_n s.t $a_n \notin (-3/2, -1/2)$.

CASE 2 $L \neq -1$. Same argument as above

Quantifiers: The defⁿ says

"$\forall \varepsilon > 0$, $\exists N$ s.t. $\forall n > N$, ..." So any proof of convergence has to start with an arbitrary ε.

For divergence, take any $L \in \mathbb{R}$ & show that $\exists \varepsilon > 0$ s.t $\forall N$, $\exists n \geq N$ s.t

$$|a_n - L| > \varepsilon.$$

- Template for convergence $\lim_{n\to\infty} a_n = L$.

1) Let $\varepsilon > 0$ be arbitrary.

2) (Scratch work) Find N that works.

3) Now show N works i.e let $n > N$ & show $|a_n - L| < \varepsilon$.

Example: 1) Show $\lim_{n\to\infty} \frac{3n+1}{7n-4}$ exists & compute it.

Pf: (Note that

$$\frac{3n+1}{7n-4} = \frac{3 + 1/n}{7 - 4/n}.$$

Claim $\lim_{n\to\infty} \frac{3n+1}{7n-4} = \frac{3}{7}$.

Pf: Let $\varepsilon > 0$. Note

$$\left| \frac{3n+1}{7n-4} - \frac{3}{7} \right| = \left| \frac{19}{7(7n-4)} \right|.$$

Aim Find N s.t

$$n > N \implies \frac{19}{7|7n-4|} < \varepsilon.$$

(Scratch). If $n \to \infty$, $7n-4 > 0$ eventually.

$$\frac{19}{7|7n-4|} < \varepsilon \iff 7n-4 > \frac{19}{7\varepsilon}$$

$$\iff n > \underbrace{\frac{4}{7} + \frac{19}{49\varepsilon}}_{N}.$$

Let $N = \frac{4}{7} + \frac{19}{49\varepsilon}$. Then

$$n > N \implies \left| \frac{3n+1}{7n-4} - \frac{3}{7} \right| < \varepsilon.$$

Since ε is arbitrary, $\lim_{n\to\infty} \frac{3n+1}{7n-4} = \frac{3}{7}$.

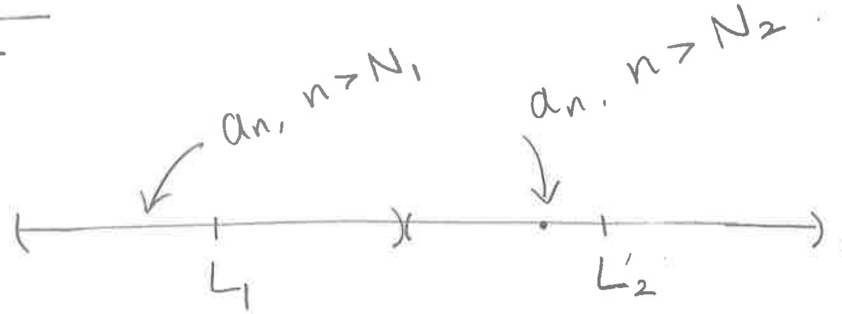
· Properties of limits

Th^m 2·1 (Uniqueness) If $a_n \xrightarrow{n\to\infty} L_1$ and $a_n \xrightarrow{n\to\infty} L_2$

Then $L_1 = L_2$

Pf: If not. Then $|L_1 - L_2| > 0$. Let

$$\varepsilon = \frac{|L_1 - L_2|}{2}$$



$\exists\ N_1$ s.t. $\forall n > N_1$, $|a_n - L_1| < \varepsilon$.

$\exists\ N_2$ s.t $\forall n > N_2$, $|a_n - L_2| < \varepsilon$.

Let $N = \max(N_1, N_2)$. Then

$\forall n > N$, $|a_n - L_1|, |a_n - L_2| < \varepsilon$.

Triangle ineq $\Rightarrow |L_1 - L_2| \le |a_n - L_1| + |a_n - L_2|$.

$$< 2\varepsilon = |L_1 - L_2|.$$

i.e $|L_1 - L_2| < |L_1 - L_2|$.

Contradiction. So $L_1 = L_2$.

Def^n: A seq^n $\{a_n\}_{n=1}^{\infty}$ is said to be <u>bounded</u> if it's set of values is bounded. i.e $\exists\ M$ s.t

$$|a_n| \le M \quad \forall n \in \mathbb{N}.$$

Th^m 2.2 If $\{a_n\}$ is convergent, then it is also bounded.

Pf: Take $\varepsilon = 1$. $\exists$ N s.t $\forall n > N$,

$$a_n \in (L-1, L+1), \text{ where } L = \lim_{n\to\infty} a_n.$$

i.e $n > N \Rightarrow |a_n| < |L| + 1$.

Let $M = \max(|a_1|, |a_2|, \dots, |a_N|, |L|+1)$.

Then $|a_n| \le M$ $\forall n \in \mathbb{N}$.

So $\{a_n\}$ is bounded.

Rk Converse is NOT true e.g $a_n = (-1)^n$ is a bounded seq^n but not convergent.

Th^m 2.3 (Alg. theorem). Let $\lim_{n\to\infty} a_n = A$, $\lim_{n\to\infty} b_n = B$.

① If $c \in \mathbb{R}$, then $\lim_{n\to\infty} c \cdot a_n = c \cdot \lim_{n\to\infty} a_n = cA$.

② $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = A + B$.

③ $\lim_{n\to\infty} a_n b_n = A \cdot B$.

④ If $a_n \ne 0$ $\forall n$, and $A \ne 0$. then

$$\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{A}.$$

and so (3) $\Rightarrow$ $\lim_{n\to\infty} \dfrac{b_n}{a_n} = \dfrac{B}{A}$.

Pf: We prove ③ & ④. ① & ② are easier.

③ Let $\varepsilon > 0$.

Aim: Est. $|a_n b_n - AB| = |a_n b_n - a_n B + a_n B - AB|$

$$\leq |a_n||b_n - B| + |a_n - A||B|. \quad (*)$$

Thm 2.2 $\Rightarrow$ $\exists\, M$ s.t. $|a_n| \leq M \ \forall n$.

Choose M big s.t. $|B| \leq M$.

Now $a_n \to A \Rightarrow \exists\, N_1$ s.t. $|a_n - A| < \frac{\varepsilon}{2M} \ \forall n > N_1$

$b_n \to B \Rightarrow \exists\, N_2$ s.t. $|b_n - B| < \frac{\varepsilon}{2M} \ \forall n > N_2$

Let $N = \max(N_1, N_2)$.

$n > N \Rightarrow |a_n||b_n - B| + |a_n - A||B|$

$$< M \cdot \frac{\varepsilon}{2M} + \frac{\varepsilon}{2M} \cdot M = \varepsilon.$$

So $(*)$ $\Rightarrow$

$n > N \Rightarrow |a_n b_n - AB| < \varepsilon$.

So $\lim_{n \to \infty} a_n b_n = AB$.

④ Let $\varepsilon > 0$.

Aim: Est. $\left|\frac{1}{a_n} - \frac{1}{A}\right| = \frac{|a_n - A|}{|a_n A|}$.

Since $A \neq 0$, $\exists\ N_1$ s.t $\forall\ n > N_1$

$|a_n| > |A|/2$.

Then for $n > N_1$, $\left|\frac{1}{a_n} - \frac{1}{A}\right| < \frac{2|a_n - A|}{|A|^2}$.

Now, $a_n \to A \Rightarrow \exists\ N_2$ s.t $n > N_2$.

$$\Rightarrow |a_n - A| < \frac{|A|^2}{2} \varepsilon.$$

Let $N = \max(N_1, N_2)$. Then

$$n > N \Rightarrow \left|\frac{1}{a_n} - \frac{1}{A}\right| < \varepsilon.$$

So $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{A}$.

<u>Th^m 2.4</u> (Squeeze Principle) Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ sequences s.t $a_n \leq b_n \leq c_n$. Then

(1) If $a_n \to A$, $b_n \to B$, $c_n \to C$ then

$$A \leq B \leq C.$$

(2) If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$. Then $\lim_{n\to\infty} b_n$ exist and is L.

<u>Pf</u> (2). Let $\varepsilon > 0$. $a_n \to L \Rightarrow \exists\ N_1$ s.t

$$|a_n - L| < \varepsilon \quad \forall\ n > N_1.$$

or $$L - \varepsilon < a_n < L + \varepsilon \quad \forall\ n > N_1$$

$C_n \to L \Rightarrow \exists\ N_2$ s.t. $\forall\ n > N_2$.

$$L - \varepsilon < C_n < L + \varepsilon$$

Let $N = \max(N_1, N_2)$.

$n > N \Rightarrow \quad b_n \geq a_n > L - \varepsilon$.

$\qquad\qquad b_n \leq C_n < L + \varepsilon$.

or $n > N \Rightarrow |b_n - L| < \varepsilon$.

Th^m 2.5 (Standard limits).

(1) $\displaystyle\lim_{n\to\infty} \frac{1}{n^p} = 0$ for $p > 0$.

(2) $\displaystyle\lim_{n\to\infty} a^n = 0$ if $|a| < 1$.

(3) $\displaystyle\lim_{n\to\infty} n^{1/n} = 1$.

(4) $\displaystyle\lim_{n\to\infty} a^{1/n} = 1$ for $a > 0$.

(5) $\displaystyle\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$, $\alpha \in \mathbb{R}$, $p > 0$.

Note: For $x > 0$, $\alpha = p/q$, define $x^\alpha = \sqrt[q]{x^p}$.

If $\alpha \in \mathbb{I}$, for any any $n > 0$, $\exists\ r_n \in \mathbb{Q}$ & $r_n \in (\alpha - 1/n, \alpha)$. Then $r_n \xrightarrow{n\to\infty} \alpha$

One can show that $x^{r_n} \to L$.

We define $L = x^\alpha$.

Pf: 1) Let $\varepsilon > 0$. Let $N = \left(\frac{1}{\varepsilon}\right)^{1/p}$. Then

$n > N \Rightarrow \quad n^p > 1/\varepsilon \quad$ or $\quad \left| \frac{1}{n^p} - 0 \right| < \varepsilon$.

2). We show $\lim\limits_{n\to\infty} |a|^n = 0$. Then since

$$-|a| \le a \le |a|$$

Squeeze th^m $\Rightarrow$ $a^n \to 0$.

Let $b = \frac{1}{|a|} - 1 > 0$ since $|a| < 1$. (i.e. $|a| = \frac{1}{1+b}$)

Then Binomial Th^m $\Longrightarrow$

$$(1+b)^n = 1 + nb + \frac{n(n-1)}{2} b^2 + \cdots + b^n$$

$$\ge 1 + nb > nb.$$

So, $|a^n| = \frac{1}{(1+b)^n} < \frac{1}{nb}$.

Given $\varepsilon > 0$, let $N = 1/b\varepsilon$. Then

$n > N \Rightarrow \quad |a^n| < \varepsilon$.

3) Let $a_n = n^{1/n} - 1$. Clearly $a_n \ge 0$.

Claim: $\lim\limits_{n\to\infty} a_n = 0$

Pf: $1 + a_n = n^{1/n}$ or $n = (1+a_n)^n$.

Binomial $\Rightarrow (1+a_n)^n = 1 + n a_n + \frac{n(n-1)}{2} a_n^2 + \cdots$

$$> \frac{n(n-1)}{2} a_n^2 \quad \text{for } n > 2$$

i.e. $$n \geq \frac{(n-1)\cdot n}{2} a_n^2$$

or $$0 \leq a_n < \sqrt{\frac{2}{n-1}} \xrightarrow{n\to\infty} 0.$$

Squeeze Th^m $\Rightarrow$ $\lim_{n\to\infty} a_n = 0.$

(4). Sps $a \geq 1$. Then for $n \geq a$,

$$1 \leq a^{1/n} \leq n^{1/n}.$$

Squeeze Th^m & (3) $\Rightarrow$ $\lim_{n\to\infty} a^{1/n} = 1.$

If $a < 1$, then let $b = 1/a \geq 1$.

$$\Rightarrow \lim_{n\to\infty} b^{1/n} = 1.$$

$$\Rightarrow \lim_{n\to\infty} a^{1/n} = \frac{1}{\lim_{n\to\infty} \frac{1}{a^{1/n}}} = 1.$$

(5) Let $k \in \mathbb{N}$ s.t. $k > \alpha$. Then for $n \geq 2k$.

Binomial theorem $\Rightarrow$

$$(1+p)^n = 1 + np + \cdots + \frac{n(n-1)\cdots(n-k+1)}{k!} p^k$$

$$> \frac{n(n-1)\cdots(n-k+1)}{k!} p^k$$

$$> \frac{n^k p^k}{2^k k!}$$

So, $0 < \dfrac{n^\alpha}{(1+p)^n} < \dfrac{2^k k!}{p^k} \cdot n^{\alpha-k} \quad (n \geq 2k)$.

$$\alpha - k < 0 \implies \lim_{n\to\infty} n^{\alpha-k} = 0.$$

$$\implies \lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0.$$

Rk 5 says that exponential grows faster than any power n^α.

· Example Compute $\lim_{n\to\infty} \dfrac{3n+1}{7n-4}$.

Sol^n: $\lim_{n\to\infty} \dfrac{3n+1}{7n-4} = \lim_{n\to\infty} \dfrac{3+1/n}{7-4/n}$.

$$= \frac{\lim_{n\to\infty}\left(3+\frac{1}{n}\right)}{\lim_{n\to\infty}\left(7-\frac{4}{n}\right)} = \frac{3}{7}$$

Monotonic Sequences

<u>Def</u>^{n}. A seq^n $\{a_n\}$ is

(1) called <u>increasing</u>, denoted by $a_n \uparrow$, if

$$a_{n+1} \geq a_n \quad \forall\ n \in \mathbb{N}.$$

(2) called <u>decreasing</u>, denoted by $a_n \downarrow$, if

$$a_{n+1} \leq a_n \quad \forall\ n \in \mathbb{N}$$

(3) called <u>monotonic</u> if increasing / decreasing.

<u>Ex</u>: $a_n = 1 - \frac{1}{n}$, $b_n = n^2$, $c_n = \left(1 + \frac{1}{n}\right)^n$, $d_n = (-1)^n/n$, $e_n = \frac{1}{n}$

It is easy to see that a_n, $b_n \uparrow$ and $e_n \downarrow$. One can also show, though this is non-trivial, that $c_n \uparrow$. So a_n, b_n, c_n, e_n are monotonic, but d_n is NOT.

<u>Th</u>^{m} 2.6 (Monotone conv. theorem): Let $\{a_n\}$ be a seq^n.

① If $a_n \uparrow$ and <u>bounded above</u>, then a_n converges

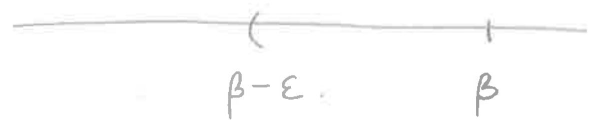
② If $a_n \downarrow$ and <u>bounded below</u>, then a_n converges

<u>Pf</u> (1). a_n bounded above $\Longrightarrow$ $\exists$ $\sup\{a_n\} \in \mathbb{R}$.

Let
$$\beta = \sup_{n=1,2,\ldots} \{a_n\}$$

<u>Claim</u>: $a_n \longrightarrow \beta$ as $n \longrightarrow \infty$.

Pf. Let $\varepsilon > 0$. Firstly $a_n \leq \beta \;\; \forall \, n \in \mathbb{N}$



$\exists \; N$ s.t $a_N > \beta - \varepsilon$. If not, then $\beta - \varepsilon$ would be an u.b for $\{a_n\}$ contradicting that β is l.u.b.
But then $a_n \uparrow \Rightarrow$

$$\forall \, n > N, \quad a_n > \beta - \varepsilon.$$

So $\forall \, n > N \quad \beta - \varepsilon < a_n \leq \beta.$

$\Rightarrow \forall \, n > N, \quad |a_n - \beta| < \varepsilon.$

$\Rightarrow a_n \longrightarrow \beta.$

(2) Apply the same arg. or directly apply (1) to the seq^n $\{-a_n\}$. Note $a_n \downarrow \Leftrightarrow -a_n \uparrow$.

<u>Cor</u> 2.7 All bounded monotonic sequences converge.

<u>Example</u> (1) Let $a_1 = 1$, & $a_{n+1} = \left(\frac{n}{n+1}\right) a_n^2$, $n \geq 1$.

$n = 1 \quad a_2 = \frac{1}{2}$,

$n = 2, \quad a_3 = \frac{2}{3} \cdot a_2^2 = \frac{1}{6}$

$\underline{n=3}$ $\quad a_4 = \frac{3}{4} a_3^2 = \frac{1}{48}$

$\underline{\text{Claim 1}}$ $\quad a_n \downarrow$ i.e $\quad a_{n+1} \leq a_n \quad \forall\ n$.

"$\underline{\text{Failed proof}}$" Try to use induction directly.

$n=1$ trivial. If $a_{n+1} \leq a_n$ for $\underline{\text{some}}$ n.

Then
$$a_{n+2} = \frac{n+1}{n+2} \cdot a_{n+1}^2 \leq \frac{n+1}{n+2} a_n^2 = \frac{(n+1)^2}{n(n+2)} a_{n+1}.$$

But $(n+1)^2/n(n+2) \geq 1$, so this does not help.

Instead we first show.

$\underline{\text{Claim 2}}$ $\quad 0 \leq a_n \leq 1 \quad \forall\ n$.

$\underline{\text{Pf of Claim 2}}$: Induction.

- $\underline{\text{Base Case}}$ $\quad n=1$. $a_1 = 1$ or $0 < a_1 \leq 1$.
- $\underline{\text{Inductive Step}}$. Sps $a_n \in (0, 1]$ for $\underline{\text{some}}$ 'n'.

Then
$$a_{n+1} = \frac{n}{n+1} a_n^2 > 0$$

Also
$$a_{n+1} = \frac{n}{n+1} a_n^2 \leq 1 \quad \text{since } \frac{n}{n+1} < 1.$$

Done!

$\underline{\text{Pf of Claim 1}}$:
$$a_{n+1} = \frac{n}{n+1} a_n^2 = \left(\frac{n}{n+1} \cdot a_n\right) a_n$$
$$\leq a_n \quad \text{since } \frac{n}{n+1} a_n < 1.$$

Now, $a_n \downarrow 0$ and $a_n > 0 \Longrightarrow \{a_n\}$ converges.

Computing limit Let $\lim_{n\to\infty} a_n = L$.

Then $\lim_{n\to\infty} a_{n+1} = L$. (homework)

Taking $\lim_{n\to\infty}$ of $a_{n+1} = \frac{n}{n+1} a_n^2$.

$$L = 1 \cdot L^2.$$

$$\Longrightarrow L^2 - L = 0$$

$$\Longrightarrow L = 0 \text{ or } 1.$$

Since $a_n \downarrow$ and $a_1 = 1$, $\boxed{L = 0}$.

Def^n: Given a seq^n $\{a_n\}_{n=1}^{\infty}$.

① We say $\{a_n\}$ diverges to ∞ if for any $M > 0$. $\exists$ $N = N(M) \in \mathbb{N}$ s.t

$$n > N \Longrightarrow a_n > M.$$

We write $\lim_{n\to\infty} a_n = \infty$.

(2) We say $\{a_n\}$ diverges to $-\infty$ if for any $M > 0$, $\exists$ $N = N(M)$ s.t.

$$n > N \Longrightarrow a_n < -M.$$

<u>Example</u> $\lim_{n\to\infty} \dfrac{3-n^2}{1+n} = -\infty$.

<u>Pf</u>: Given $M > 0$ we need to find an N that works.

Note $\dfrac{3-n^2}{1+n} = -\dfrac{n^2-3}{1+n} = -n\left(\dfrac{1-3/n^2}{1+1/n}\right)$.

If $n > 3$, $1 - 3/n^2 \geq \frac{1}{2}$; $0 < 1 + \frac{1}{n} \leq 2$

So $n > 3 \Rightarrow -n\left(\dfrac{1-3/n^2}{1+1/n}\right) < -\dfrac{n}{2}$.

Given M, let $N = \max(2M, 3)$. Then

$$n > N \Rightarrow n > 2M,\ n > 3$$

$$\Rightarrow -\frac{n}{2} < -M,\ n > 3$$

$$\Rightarrow \frac{3-n^2}{1+n} < -M.$$

<u>Th^m 2.8</u> (1) An increasing unbounded seq^n diverges to ∞.

(2) A decreasing unbounded seq^n diverges to $-\infty$.

<u>Pf</u> (1) If $\{a_n\}$ is increasing & unbounded

$$\sup a_n = \infty.$$

Given M, $\exists$ N s.t $a_N > M$.

$a_n \uparrow \Rightarrow \forall n > N, \quad a_n > M$

So $\lim_{n\to\infty} a_n = \infty$.

(2) Similar argument.

- Bolzano - Weierstrass

Def^n Suppose $\{a_n\}$ is a sequence, and $\{n_k\}$ is a sequence of numbers in $\mathbb{N}$, we then call $\{a_{n_k}\}$ a subsequence of $\{a_n\}$.

Alternatively, a subsequence is another seq^n $\{b_k\}_{k=1}^{\infty}$ s.t for each $k \in \mathbb{N}$, $\exists$ n_k s.t

(1) $n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$

(2) $b_k = a_{n_k} \quad \forall k$.

Rk: One can simply choose $n_k = k \ \forall k$ so that $\{a_n\}$ is a sub-seq^n of itself.

Example: Let $a_n = (-1)^n$. Then $\{a_n\}$ diverges.

Consider
$$b_k = a_{2k} = \{1, 1, \cdots \}.$$

Then $\{b_k\}$ is a sub-sequence which converges.

Recall that any convergent seq^n is bounded but the converse is false. e.g $a_n = (-1)^n$.
Our main theorem for this section is a partial converse. First we have the foll.

Th^m 2.9 (If $\{a_n\}$ is a seq^n s.t. $a_n \to L$. Then every subseq^n $a_{n_k} \to L$.

Pf: Given $\varepsilon > 0$. Let $b_k = a_{n_k}$.

$a_n \to L \implies \exists\ N$ s.t

$$n > N \implies |a_n - L| < \varepsilon.$$

Since $n_1 < n_2 < \cdots < n_k < \cdots$

$$n_k \longrightarrow \infty \text{ as } k \to \infty.$$

In particular $\exists\ K$ s.t

$$k > K \implies n_k > N.$$

Then if $k > K \implies |a_{n_k} - L| < \varepsilon$.

$$\implies |b_k - L| < \varepsilon.$$

$$\implies \lim_{k\to\infty} b_k = L.$$

Th^m 2.10 (Bolzano - Weierstrass). Every bounded seq^n has a convergent subsequence.

We first need the foll.

<u>Lemma 2.11</u> Every seq^n has a monotonic subsequence.

<u>Pf of Th^m 2.10</u>: Let $\{a_n\}$ be a seq^n. By Lem 2.11 $\exists$ sub-seq^n $\{a_{n_k}\}$ which is monotonic.
$\{a_n\}$ bounded $\Longrightarrow$ $\{a_{n_k}\}$ bounded.
Monotone Conv. theorem (Th^m 2.6) $\Rightarrow \{a_{n_k}\}$ converges.

<u>Pf of Lemma 2.11</u> Let $\{a_n\}$ be any seq^n
Call a_n <u>dominant</u> if

$$a_n > a_m \quad \forall \quad m > n. \qquad (*)$$

<u>CASE 1</u> $\exists$ infinitely many dominant terms
$\{a_{n_1}, a_{n_2}, a_{n_3}, \ldots\}$ with $n_1 < n_2 < n_3 < \cdots < n_k < \cdots$

$$(*) \Longrightarrow \qquad a_{n_{k+1}} < a_{n_k} \quad \forall \quad k.$$

$\Longrightarrow$ $\{a_{n_k}\}$ is a decreasing sub-seq^n.

<u>CASE 2</u> Sps $\exists$ only finitely many dominant terms. Let n_1 s.t. no dominant term a_n for $n \geqslant n_1$. Since a_{n_1} is <u>not</u> dominant, $\exists$ $n_2 > n_1$ s.t. $\qquad a_{n_2} \geqslant a_{n_1}$.

Since a_{n_2} not dominant, $\exists$ $n_3 > n_2$ s.t

$$a_{n_3} \geqslant a_{n_2}.$$

Having chosen, $a_{n_1}, \ldots, a_{n_k}$ let n_{k+1} s.t. $n_{k+1} > n_k$ but $a_{n_{k+1}} \geqslant a_{n_k}$.

Then we have a sub-seq^n $\{a_{n_k}\} \uparrow$.

In both cases we have found a monotonic sub-sequence.