Completeness

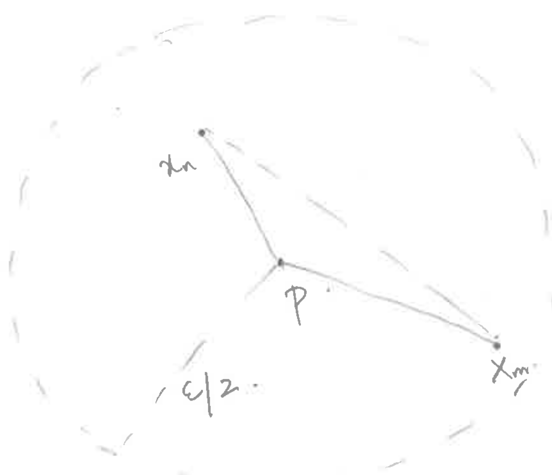
- <u>Def^n</u> Let (X, d) metric space. A seq^n $\{x_n\}$ is called <u>Cauchy</u> if $\forall\ \varepsilon > 0$, $\exists\ N$ s.t

$$n, m > N \implies d(x_n, x_m) < \varepsilon.$$

<u>Th^m 8.21</u> $\{x_n\}$ convergent $\implies$ $\{x_n\}$ Cauchy.

<u>Pf</u>: Sps $x_n \longrightarrow p$. Let $\varepsilon > 0$. Then $\exists\ N$ s.t

$$n > N \implies d(x_n, p) < \frac{\varepsilon}{2}.$$



Then if $n, m > N$

$$d(x_n, x_m) \le d(x_n, p) + d(x_m, p)$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So $\{x_n\}$ is Cauchy.

Ques: Is the converse true?

Ans: NO! in general. e.g Consider $(\mathbb{Q}, |\cdot|)$.

Let $\{q_n\}$ be the "decimal approximations" to $\sqrt{2}$.

i.e $q_1 = 1$, $q_2 = 1.4$, ..., q_n. Then

$$q_{n+1} = q_n + \frac{d_{n+1}}{10^{n+1}},$$

where d_{n+1} is the largest integer s.t. $q_{n+1} < \sqrt{2}$.

Then $q_n \in \mathbb{Q}$.

Claim 1: $\{q_n\}$ is Cauchy.

Pf: If $m > n > N$, then since each $d_k \leq 9$

$$|q_m - q_n| = \left| \frac{d_{n+1}}{10^{n+1}} + \cdots \frac{d_m}{10^m} \right|$$

$$\leq \frac{9}{10^{n+1}} \left(1 + \frac{1}{10} + \cdots \frac{1}{10^{m-n-1}} \right)$$

$$\leq \frac{9}{10^{n+1}} \left(1 + \frac{1}{10} + \frac{1}{10^2} + \cdots \right)$$

$$= \frac{9}{10^{n+1}} \cdot \frac{1}{1 - 1/10} = \frac{1}{10^n} < \frac{1}{10^N}$$

If we choose $N > -\log_{10} \varepsilon$ then $\forall\, m > n > N$

$$|q_m - q_n| < \varepsilon.$$

Claim 2: $\{q_n\}$ is not convergent in $\mathbb{Q}$.

Pf: $q_n \uparrow$. So if $q_n \longrightarrow q \in \mathbb{Q}$, then

$$q = \sup\{q_n\} = \sqrt{2} \text{ by construction}$$

But $\sqrt{2} \notin \mathbb{Q}$. Contradiction.

Def^n: A metric space is called complete if every Cauchy sequence also converges.

Th^m 8.22: $(\mathbb{R}^k, |\cdot|)$ is complete.

Pf: Let $\{\vec{X}_n\}$ be a Cauchy seq^n in $\mathbb{R}^k$.

Claim: $\{\vec{X}_n\}$ is bounded.

Pf: $\exists$ N s.t $\forall$ $m > N$

$$|\vec{X}_m - \vec{X}_N| < 1.$$

Let $M = \max(|\vec{X}_N - \vec{X}_1|, \dots, |\vec{X}_N - \vec{X}_{N-1}|, 1)$

Then clearly $\{X_n\}$ lies in $B_M(\vec{X}_N)$

So $\{\vec{x}_n\}$ is bounded.

Then, by Bolzano-Weierstrass $\exists$ sub-sequence $\{\vec{x}_{n_j}\}$ s.t $\vec{x}_{n_j} \longrightarrow \vec{p} \in \mathbb{R}^k$.

<u>Claim</u> $\vec{x}_n \longrightarrow \vec{p}$.

<u>Pf</u>: $\{\vec{x}_n\}$ Cauchy $\Longrightarrow$ given $\varepsilon > 0$, $\exists$ N s.t

$$n, m > N \Longrightarrow |\vec{x}_n - \vec{x}_m| < \varepsilon/2 \quad (*).$$

$\vec{x}_{n_j} \longrightarrow \vec{p} \Longrightarrow \exists$ J s.t $\forall$ $j > J$

$$|\vec{x}_{n_j} - \vec{p}| < \varepsilon/2 \quad (**).$$

$n_j \xrightarrow{j \to \infty} \infty$, so $\exists$ $j_0 > J$ s.t $n_{j_0} > N$.

Then $\forall$ $n > N$.

$$|\vec{x}_n - \vec{p}| \le |\vec{x}_n - \vec{x}_{n_{j_0}}| + |\vec{x}_{n_{j_0}} - \vec{p}|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So $\vec{x}_n \longrightarrow \vec{p}$.

<u>Rk</u>: In the course of the proof, we have actually proved the foll. useful facts.

Th^m 8.23 Let (X,d) be any metric space, and let $\{X_n\}$ be a Cauchy seq^n. Then

① $\{X_n\}$ is bounded.

② If $\exists$ sub-seq^n $\{X_{n_k}\}$ s.t $X_{n_k} \longrightarrow p$, then

$X_n \longrightarrow p$

• Compactness and Completeness.

Th^m 8.24 Let (X,d) be compact. Then (X,d) is also complete

Pf: Let $\{X_n\}$ be Cauchy. (X,d) compact $\Longrightarrow \exists$ subsequence s.t $X_{n_k} \longrightarrow p \in X$.

Th^m 8.23 (2) $\Longrightarrow$ $X_n \longrightarrow p$.

So every Cauchy seq^n converges.

Rk 1) Converse not true. e.g $(\mathbb{R}, |\cdot|)$ is complete but non-compact.

2) We saw in general closed + bounded $\not\Longrightarrow$ compact

e.g $(\mathbb{Q}, |\cdot|)$ and $E = \{q \in \mathbb{Q} \mid 2 < q^2 < 3\}$ is closed & bounded in $(\mathbb{Q}, |\cdot|)$ but not compact

May be problem is that $\mathbb{Q}$ is not complete.

Ques: If (X,d) complete & $K \subset X$ closed & bounded

Is K compact ?

<u>Ans</u> NO1 Consider

$$C^0[0,1] = \{ f : [0,1] \to \mathbb{R} \mid f \text{ cont. on } [0,1] \}.$$

with the metric

$$d(f,g) = \sup_{t \in [0,1]} |f(t) - g(t)|.$$

<u>Fact</u>: $(C^0[0,1], d)$ is a <u>complete</u> <u>metric</u> <u>space</u>.
essentially since uniform Cauchy $\Rightarrow$ uniform convergence.

Let

$$K = \{ f \in C^0[0,1] \mid |f(t)| \le 1 \}.$$

<u>Claim 1</u> K is closed and bounded.

<u>Pf</u>: If 0 is the zero-function. Then

$$f \in K \iff d(0,f) = \sup_{t \in [0,1]} |f(t)| \le 1.$$

So $K = B_{\le 1}(0)$. & hence is bounded & closed.

<u>Claim 2</u> K is not compact.

<u>Pf</u>: Consider $f_n \in K$ given by

$$f_n(t) = t^n.$$

Then <u>no</u> sub-sequence of f_n converges uniformly on $[0,1]$. So $\nexists$ f_{n_k} s.t $\{f_{n_k}\}$ converges in

$(C^0[0,1], d)$. So K is non compact.

For a converse, we need the foll. definition

<u>Def^n</u>: Let (X,d) any metric space. Let $E \subset X$. We say E is <u>totally bounded</u> if $\forall\, \varepsilon > 0$, $\exists$ $N = N(\varepsilon)$ and points $p_1, \cdots, p_N \in E$ s.t

$$E \subset \bigcup_{j=1}^{N} B_\varepsilon(p_j).$$

(X,d) is called a totally bounded metric space if $X \subseteq X$ is totally bounded.

<u>Rk</u> Totally bounded $\Longrightarrow$ bounded. (prove it!) but converse not true. e.g (X, d_{dist}) s.t X is an infinite set. Then X is bounded since

$$d_{dist}(p,x) \leq 1 \quad \forall\ p, x \in X.$$

but not totally bounded since if $\varepsilon < 1$. Then no matter what N is & what points $p_1, \cdots, p_N$ are picked, $X \not\subseteq \bigcup_{i=1}^{N} B_\varepsilon(p_i) = \{p_1, p_2, \cdots, p_N\}$. since X is infinite.

<u>Th^m 8.25</u>: (X,d) compact $\Longleftrightarrow$ (X,d) is complete and totally bounded.

__Cor__ 8.26. Let (X,d) be complete. Then $K \subset X$ is compact $\iff$ K is closed & totally bounded.

__Pf__: $\Longrightarrow$ K compact $\Rightarrow$ K is closed. Sps K not totally bounded. Then $\exists\ \varepsilon > 0$ s.t K is not contained in a finite union of any ε-balls. Let $p_1 \in K$. Let Then $K \not\subset B_\varepsilon(p_1)$. Let $p_2 \in K \setminus B_\varepsilon(p_1)$. Then $K \not\subset B_\varepsilon(p_1) \cup B_\varepsilon(p_2)$.

Let $p_3 \in K \setminus B_\varepsilon(p_1) \cup B_\varepsilon(p_2)$. Having chosen $p_1 \cdots, p_{n-1}$, let $p_n \in K \setminus B_\varepsilon(p_1) \cup \cdots \cup B_\varepsilon(p_{n-1})$.

Then $\{p_n\}$ is a seq in K s.t $d(p_n, p_m) \geq \varepsilon$. $\forall n, m$. So there is no sub-seq s.t $\{p_{n_k}\}$ converges. Contradiction since K compact.

__$\Longleftarrow$__. K closed $\Longrightarrow$ (K, d_K) is also complete. & totally bounded.

$Th^m \Longrightarrow K$ is compact.

- Construction of $\mathbb{R}$ as completition of $\mathbb{Q}$

__Def^n__: Let (X, d_X) and (Y, d_Y) be metric spaces & $\phi\ X \to Y$. We say.

(1) ϕ is <u>distance</u> <u>preserving</u> if

$$d_Y(\phi(x_1), \phi(x_2)) = d_X(x_1, x_2).$$

(2) ϕ is an <u>embedding</u> if it is distance preserving AND injective.

(3) ϕ is an <u>isometry</u> if it is a surjective embedding.

<u>Rk</u>: (1) Distance preserving maps are automatically cont.

(2). If ϕ is an embedding we can think of (X, d_X) as a subspace of (Y, d_Y).

(3) If $\phi: X \longrightarrow Y$ an isometry then $\phi^{-1}: Y \to X$ exist & is an isometry.

<u>Thm</u> 8.2.6. Let (X, d) be a metric space. Then $\exists$ a <u>unique</u> (upto isometries) <u>complete</u>, <u>metric sp</u> (X', d') and an embedding $\phi: X \longrightarrow X'$ s.t $\overline{\varphi(X)} = X'$ (i.e $\phi(X)$ is dense in X').

<u>Defn</u>. (X', d') is called the <u>completition</u> of (X, d).

<u>Def^n</u>. We define $\mathbb{R}$ to be the <u>completition</u> of $(\mathbb{Q}, |\cdot|)$.

<u>FACT</u>: This gives us the same set of real #s as before.