Practice problems for the final

1. Show the following using just the definitions (and no theorems).

   (a) The function
$$f(x) = \begin{cases} x^2, & x \geq 0 \\ 0, & x < 0, \end{cases}$$
   is differentiable on $\mathbb{R}$, while $f'(x)$ is not differentiable at 0.

Solution: For $x > 0$ and $x < 0$, the function is clearly differentiable. We check at $x = 0$. The difference quotient is
$$\varphi(x) = \frac{f(x) - f(0)}{x} = \frac{f(x)}{x}.$$
So
$$\varphi(0+) = \lim_{x \to 0^+} \varphi(x) = \lim_{x \to 0^+} \frac{x^2}{x} = 0,$$
$$\varphi(0-) = \lim_{x \to 0^-} \varphi(x) = \lim_{x \to 0^-} \frac{0}{x} = 0.$$
Since $\varphi(0+) = \varphi(0-)$, $\lim_{x \to 0} \varphi(x) = 0$ and hence f is differentiable at 0 with $f'(0) = 0$. So together we get
$$f'(x) = \begin{cases} 2x, & x > 0 \\ 0, & x \leq 0. \end{cases}$$
$f'(x)$ is again clearly differentiable for $x > 0$ and $x < 0$. To test differentiability at $x = 0$, we consider the difference quotient
$$\psi(x) = \frac{f'(x) - f'(0)}{x} = \frac{f'(x)}{x}.$$
Then it is easy to see that $\psi(0+) = 2$ while $\psi(0-) = 0$. Since $\psi(0+) \neq \psi(0-)$, $\lim_{x \to 0} \psi(x)$ does not exist, and f' is not differentiable at $x = 0$.

   (b) The function $f(x) = x^3 + x$ is continuous but not uniformly continuous on $\mathbb{R}$.

Solution: We show that f is continuous at some $x = a$. That is given $\varepsilon > 0$ we have to find a corresponding δ.

$|f(x) - f(a)| = |x^3 + x - a^3 - a| = |(x - a)(x^2 + xa + a^2) + (x - a)| = |x - a||x^2 + xa + a^2 + 1|.$

Now suppose we pick $\delta < 1$, then $|x - a| < \delta \implies |x| < |a| + 1$, and so
$$|x^2 + xa + a^2 + 1| \leq |x|^2 + |x||a| + a^2 + 1 \leq 3(|a| + 1)^2.$$
To see the final inequality, note that $|x||a| < (|a| + 1)|a| < (|a| + 1)^2$, and $a^2 + 1 < (|a| + 1)^2$. Combining with the first equation, we see that
$$|f(x) - f(a)| < 3(|a| + 1)^2|x - a|.$$

Now given any $\varepsilon > 0$, if

$$|x-a| < \varepsilon/3(|a|+1)^2,$$

then

$$|f(x)-f(a)| < \varepsilon.$$

But we also needed $|x-a| < 1$ in the argument. So define

$$\delta = \min\Big(1, \frac{\varepsilon}{3(|a|+1)^2}\Big).$$

Next, we show that the continuity is not uniform on $\mathbb{R}$. That is we need to show that there exists $\varepsilon > 0$ such that for any n, there exists points x_n and y_n such that

$$|x_n - y_n| \le \frac{1}{n}, \text{ but } |f(x_n)-f(y_n)| > \varepsilon.$$

Simply choose $\varepsilon = 1$, $y_n = n$ and $x_n = n + 1/n$. Then

$$|f(x_n)-f(y_n)| = \Big(n+\frac{1}{n}\Big)^3 - n^3 + \frac{1}{n} = 3n + \frac{4}{n} + \frac{1}{n^3} > 3n > 1.$$

(c) The sequence of functions

$$f_n(x) = \frac{nx}{n+1}$$

converges point wise on $\mathbb{R}$, uniformly on bounded intervals (a,b), but does NOT converge uniformly on $\mathbb{R}$.

Solution: We show that $f_n(x) \to x$ point-wise on $\mathbb{R}$.

$$|f_n(x) - x| = \Big|\frac{nx}{n+1} - x\Big| = \frac{|x|}{n+1}.$$

At $x = 0$, $f_n(0) = 0$ for all n, and so there is nothing to prove. If $x \neq 0$, given any $\varepsilon > 0$, we can pick N large enough so that $|x|/(N+1) < \varepsilon$. THen for $n > N$ we see that

$$|f_n(x) - x| < \varepsilon.$$

Of course the N depends on x and hence we have only proved point-wise. But notice that if $x \in (a,b)$ in a bounded interval, then there is an M such that $|x| < M$.Then we can simply choose $N > M/\varepsilon$, then for any $n > N$ and $x \in (a,b)$, it is easy to see that

$$|f_n(x) - x| < \varepsilon,$$

and hence the convergence is uniform on bounded intervals.

We now claim that the convergence is NOT uniform on all of $\mathbb{R}$. That is we need to show that there exists $\varepsilon > 0$, a subsequence $n_k \to \infty$ and points $x_{n_k} \in \mathbb{R}$ such that

$$|f_{n_k}(x_{n_k}) - x_{n_k}| > \varepsilon.$$

For this, let $\varepsilon = 1$, $n_k = k$ and $x_k = k+2$. Then by the calculation above

$$|f_k(x_k) - x_k| = \frac{|x_k|}{k+1} = \frac{k+2}{k+1} > 1.$$

2. Let $f : (0, 1) \to \mathbb{R}$ be a differentiable function such that $|f'(t)| \leq 1$ for all t. Show that the sequence $a_n = f(1/n)$ converges.

> **Solution:** My mean value theorem, since $|f'(t)| \leq 1$, we see that
> $$|a_n - a_m| \leq \left|\frac{1}{n} - \frac{1}{m}\right|.$$
> But since $\{1/n\}$ converges to zero, it is in particular a Cauchy sequence. This shows that the sequence $\{a_n\}$ is Cauchy, and hence must converge.

3. Let
$$f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0, & \text{otherwise.} \end{cases}$$

(a) Compute the upper and lower integrals on $[0, 1]$.

> **Solution:** Let $\mathcal{P} = \{t_0, t_1, \cdots, t_n\}$ be any partition of $[0, 1]$. Recall that
> $$U(\mathcal{P}, f) = \sum_{k=1}^{n} M_k(t_k - t_{k-1}),$$
> $$L(\mathcal{P}, f) = \sum_{k=1}^{n} m_k(t_k - t_{k-1}),$$
> where
> $$M_k = \sup_{[t_{k-1}, t_k]} f(t), \ m_k = \inf_{[t_{k-1}, t_k]} f(t).$$
> Clearly $L(\mathcal{P}, f) = 0$ for any partition $\mathcal{P}$, and hence $L(f) = \sup_{\mathcal{P}} L(\mathcal{P}, f) = 0$. For the upper integral, consider the function $g(x) = x$ on $[0, 1]$. Since rationals are dense in $[0, 1]$ and $x \geq 0$, it follows that $U(\mathcal{P}, f) = U(\mathcal{P}, g))$ for all partitions. Hence $U(f) = U(g)$. But since g is integrable on $[0, 1]$, it follows that
> $$U(g) = \int_0^1 x \, dx = \frac{1}{2}.$$
> Hence $U(f) = 1/2$.

(b) Now do the same for the interval $[-1, 1]$.

> **Solution:** Let
> $$g(x) = \begin{cases} 0, & x \in [0, 1] \\ x, & x \in [-1, 0) \end{cases}, \ h(x) = \begin{cases} x, & x \in [0, 1] \\ 0, & x \in [-1, 0) \end{cases}.$$
> **Claim.** For any partition $\mathcal{P}$ of $[-1, 1]$,
> $$L(\mathcal{P}, f) = L(\mathcal{P}, g), \ U(\mathcal{P}, f) = U(\mathcal{P}, h).$$
> **Proof.** We prove the first equality, the second one is similar. Let $\mathcal{P} = \{t_0, \cdots, t_n\}$ be a partition of $[-1, 1]$. Let $m_k(f)$ and $m_k(g)$ be the infimums of f and g respectively on $[t_{k-1}, t_k]$. Let $l \in \{1, 2 \cdots, n\}$ such that $0 \in (t_{l-1}, t_l]$. Then for all $k \leq l$, $m_k(f) = m_k(g) = t_{k-1}$ and for all $k > l$, $m_k(f) = m_k(g) = 0$. In either case $m_k(f) = m_k(g)$ and hence $L(\mathcal{P}, f) = L(\mathcal{P}, g)$. $\square$

It follows from the claim that $L(f) = L(g)$ and $U(f) = U(h)$. Since $g, h \in \mathcal{R}[-1,1]$, we have

$$L(g) = \int_{-1}^{1} g(x)\,dx = \int_{-1}^{0} x\,dx = -\frac{1}{2},$$

and

$$U(h) = \int_{-1}^{1} h(x)\,dx = \int_{0}^{1} x\,dx = \frac{1}{2}.$$

Hence $L(f) = -1/2$ and $U(f) = 1/2$.

4. Suppose f is a continuous real valued function on $[0,\infty)$ which is differentiable on $(0,\infty)$ and satisfies

$$f'(t) > f(t)$$

for all $t \in (0,\infty)$. If $f(0) = 1$, show that $f(t) > e^t$ for all t.

Solution: Let $g(t) = e^{-t}f(t)$. By the hypothesis, g is differentiable, and

$$g'(t) = e^{-t}(f'(t) - f(t)) > 0.$$

So $g(t)$ is an increasing function. Since $g(0) = f(0) = 1$, $g(t) > 1$ for all $t > 0$. That is, $f(t) > e^t$ for all $t > 0$.

5. For any $\vec{x} = (x_1, x_2)$ and $\vec{y} = (y_1, y_2)$ in $\mathbb{R}^2$, define

$$d(\vec{x}, \vec{y}) = \max(|x_1 - y_1|, |x_2 - y_2|).$$

(a) Show that $(\mathbb{R}^2, d)$ is a metric space.

Solution: Positive definiteness and symmetry are trivial. We only need to show the triangle inequality. It is enough to prove the tirangle inequality for the triple $\vec{x}$, $\vec{y}$ and $\vec{0}$. That is, we need to prove

$$\max(|x_1 - y_1|, |x_2 - y_2|) \le \max(|x_1|, |x_2|) + \max(|y_1|, |y_2|). \tag{0.1}$$

Let $M = \max(|x_1|, |x_2|)$ and $L = \max(|y_1|, |y_2|)$. By the triangle inequality for real numbers,

$$|x_1 - y_1| \le |x_1| + |y_1| \le M + L.$$

Similarly, $|x_2 - y_2| \le M + L$, and this proves the inequality in (0.1).

(b) Show that

$$\frac{|\vec{x} - \vec{y}|}{\sqrt{2}} \le d(\vec{x}, \vec{y}) \le |\vec{x} - \vec{y}|.$$

Solution: Let $M = d(x, y)$. Then $|x_1 - y_1| \le M$ and $|x_2 - y_2| \le M$. But then

$$|\vec{x} - \vec{y}| = \sqrt{(x_1 - y_2)^2 + (x_2 - y_2)^2} \le \sqrt{M + M} = M\sqrt{2}.$$

This shows the first inequality. The second follows from the observation that $|\vec{x} - \vec{y}| \ge |x_1 - y_1|, |x_2 - y_2|$ and hence $|\vec{x} - \vec{y}| \ge d(\vec{x}, \vec{y})$.

(c) Hence, show that $(\mathbb{R}^2, d)$ is a complete metric space.

> **Solution:** From part(b), it follows that a sequence $\vec{x_n}$ converges to $\vec{p}$ in the metric d if and only if it converges in the Euclidean metric. Similarly, the sequence is Cauchy in the metric d if and only if it is Cauchy in the Euclidean metric. Since $\mathbb{R}^2$ with the Euclidean metric is complete, it then follows that $(\mathbb{R}^2, d)$ is complete.

(d) Also show that a subset K is compact in $(\mathbb{R}^2, d)$ if and only if it is closed and bounded.

> **Solution:** Similar to part(d). Write up a complete proof on your own.

6. Consider the rectangle $R \subset \mathbb{R}^2$ formed by the edges $x = \pm 2$, $y = 0$ and $y = 1$ (here we consider only the boundary rectangle, and not the interior). We can think of R as a metric space with the metric induced from $\mathbb{R}^2$.

(a) Describe the sets $B_1(\vec{0})$ and $B_{\leq 1}(\vec{0})$.

> **Solution:** The sets
>
> $$\overline{B_1(\vec{0})} = \{(x, 0) \mid -1 \leq x \leq 1\}$$
> $$B_{\leq 1}(\vec{0}) = \{(x, 0) \mid -1 \leq x \leq 1\} \cup \{(0, 1)\},$$
>
> and so clearly the two sets are not equal.

(b) Show that $\overline{B_1(\vec{0})} \neq B_{\leq 1}(\vec{0})$.

> **Solution:** The point $(0, 1)$ belongs to $B_{\leq 1}(\vec{0})$ but not $\overline{B_1(\vec{0})}$ and so the two sets are not equal.

(c) On the other hand, show that for any metric space (X, d),

$$\overline{B_r(p)} \subseteq B_{\leq r}(p).$$

> **Solution:** We need to show that for any $q \in \overline{B_r(p)}$, $d(p, q) \leq r$, and hence $q \in B_{\leq r}(p)$. We argue by contradiction. Suppose there is a $q \in \overline{B_r(p)}$ with $d(p, q) > r$. Let $\varepsilon > 0$ such that $r + \varepsilon < d(p, q)$, and consider the ball $B_\varepsilon(q)$. Let $x \in B_\varepsilon(q)$. Then
>
> $$d(p, x) \geq d(p, q) - d(q, x) \geq d(p, q) - \varepsilon > r.$$
>
> This shows that $B_\varepsilon(q)$ does not intersect $B_r(p)$, a contradiction since q is a limit point of $B_r(p)$.

7. Let (X, d) be a metric space. As usual, denote by $\overline{S}$ and $\text{int}(S)$, the closure and interior of a subset S respectively. Let $\mathcal{F}$ be an arbitrary collection of subsets.

(a) Show that

$$\cup_{A \in \mathcal{F}} \text{int}(A) \subseteq \text{int}(\cup_{A \in \mathcal{F}} A).$$

> **Solution:** Let $p \in \cup_{A \in \mathcal{F}} \text{int}(A)$. Then $p \in \text{int}(A)$ for some $A \in \mathcal{F}$. SO there exists an $r > 0$ such that $B_r(p) \subset A \subset \cup_{A \in \mathcal{F}} A$. Hence p is also an interior point of $\cup_{A \in \mathcal{F}} A$.

(b) Give an example of a metric space (X, d) and a finite collection of subsets $\mathcal{F}$ for which equality does not hold.

> **Solution:** Let $X = \mathbb{R}$ with the usual Euclidean metric. Consider the sets $A_1 = (0, 1]$ and $A_2 = [1, 2)$. Then $\mathrm{int}(A_1) \cup \mathrm{int}(A_2) = (0, 2) \setminus \{1\}$, but $\mathrm{int}(A_1 \cup A_2) = (0, 2)$.

(c) Show that

$$\cup_{A \in \mathcal{F}} \overline{A} \subseteq \overline{\cup_{A \in \mathcal{F}} A}.$$

> **Solution:** Let $p \in \cup_{A\in\mathcal{F}} \overline{A}$. Then $p \in \overline{A}$ for some $A \in \mathcal{F}$. So for any $r > 0$, $B_r(p) \cap A \neq \phi$. But then $B_r(p) \cap (\cup_{A\in\mathcal{F}} A)$ is non-empty. SInce this is true for any $r > 0$, $p \in \overline{\cup_{A\in\mathcal{F}} A}$

(d) Give an example of a metric space (X, d) and a countable collection of subsets $\mathcal{F}$ for which equality does not hold above.

> **Solution:** Again let X be $\mathbb{R}$ with the Euclidean metric, and let $A_k = (1/k, 1)$, for $k = 1, 2, \cdots$. Then
>
> $$\cup_{k=1}^{\infty} \overline{A_k} = (0, 1]$$
> $$\overline{\cup_{k=1}^{\infty} A_k} = [0, 1],$$
>
> and clearly the first set is strictly smaller than the second one.

(e) Show that if $\mathcal{F}$ is a *finite* collection of sets, then

$$\cup_{A \in \mathcal{F}} \overline{A} = \overline{\cup_{A \in \mathcal{F}} A}.$$

> **Solution:** Let $\mathcal{F} = \{A_1, \cdots, A_N\}$ be the finite collection os sets. From part(c),
>
> $$\cup_{k=1}^{N} \overline{A_k} \subseteq \overline{\cup_{k=1}^{n} A_k}.$$
>
> For the reverse inclusion, let $p \in \overline{\cup_{k=1}^{n} A_k}$. Suppose $p \notin \overline{A_k}$ for all k. Then for each k, there exists $r_k > 0$ such that $B_{r_k}(p) \cap A_k = \phi$. Let
>
> $$r := \min(r_1, \cdots, r_N).$$
>
> Then $B_r(p) \subset B_{r_k} A_k$, and so $B_r(p) \cap A_k = \phi$ for all k. This implies that
>
> $$B_r(p) \cap \cup_{k=1}^{N} A_k = \phi,$$
>
> and so $p \notin \overline{\cup_{k=1}^{n} A_k}$, a contradiction. So there is at least one k such that $p \in \overline{A_k}$, and so $p \in \cup_{k=1}^{N} \overline{A_k}$. This shows that
>
> $$\overline{\cup_{k=1}^{n} A_k} \subseteq \cup_{k=1}^{N} \overline{A_k},$$
>
> and hence the two sets must be equal.

8. A point $p \in X$ is called a *fixed point* of a map $f : X \to X$ if $f(p) = p$.

(a) If (X, d) is a compact metric space, and f satisfies

$$d(f(x), f(y)) < d(x, y),$$

for $x \neq y$, show that f has a fixed point in X and that fixed point is unique. **Hint.** Consider the minimum of $d(x, f(x))$.

Solution: Let $g(x) = d(x, f(x))$. Then g is a continuous function on X. Let

$$m = \inf_{x \in X} g(x).$$

Since $g(x) \geq 0$, note that $m \geq 0$. Now, since X is compact, by the extremum value theorem, the minimum is attained. So there exists a $p \in X$ such that $m = d(p, f(p))$.

Claim. $m = 0$ and hence $f(p) = p$. That is, p is a fixed point.

Proof. If not, then $m > 0$. In particular, $f(p) \neq p$. But then since f decreases distances,

$$g(f(p)) = d(f(p), f(f(p))) < d(p, f(p)) = m.$$

That is, $f(p)$ decreases the value of $g(x)$ further, which is a contradiction since m is the infimum. Hence $m = 0$, completing the proof of the claim and the problem.

(b) Show that the statement is no longer true if X is merely assumed to be complete, by considering the following example - $f : (-\infty, \infty) \to \mathbb{R}$ given by

$$f(t) = t + \frac{1}{1 + e^t}.$$

Hint. Show that $0 < f'(t) < 1$ for all t.

Solution: By quotient rule one sees that

$$f'(t) = 1 - \frac{e^t}{(1 + e^t)^2} < 1.$$

On the other hand since $1 + e^t > e^t$, we see that

$$f'(t) > 1 - \frac{e^t}{e^{2t}} > 1 - e^{-t} > 0$$

since $t > 0$. Then by hte mean value theorem for any $s > t$, there is a point $c \in (t, s)$ such that

$$f(s) - f(t) = f'(c)(s - t).$$

Taking absolute values we see that

$$|f(s) - f(t)| < |s - t|.$$

Hence f satisfies the hypothesis in part(a), but we claim that f cannot have a fixed. Since if p is a fixed point then

$$p = f(p) = p + \frac{1}{1 + e^p},$$

or equivalently

$$\frac{1}{1 + e^p} = 0,$$

which is impossible. This shows that compactness was necessary in part(a), and that the conclusion of part(a) is no longer valid if we only assume X (which in the present case is $(0, \infty)$) is complete.

(c) If X is complete, $f : X \to X$, and

$$d(f(x), f(y)) \leq \alpha d(x, y)$$

for some $\alpha < 1$, then show that there is a unique fixed point in X. **Hint.** Let $x_0 \in X$ be any point, and define $x_{n+1} = f(x_n)$. , and show that if $m > n$,

$$d(x_m, x_n) \leq \frac{\alpha^n d(x_1, x_0)}{1 - \alpha}.$$

Solution: As the hint suggests, pick any $x_0 \in X$ and let $x_{n+1} = f(x_n)$. We first show that $\{x_n\}$ is a Cauchy sequence. The method of proof is the same as in the final problem in the first mid-term. We first note that for any $k > 0$,

$$d(x_{k+1}, x_k) = d(f(x_k), f(x_{k-1})) \leq \alpha d(x_k, x_{k-1}) \leq \alpha^2 d(x_{k-1}, x_{k-2}) \leq \cdots \leq \alpha^k d(x_1, x_0).$$

For any $m > n$ using triangle inequality, we then estimate,

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n) \\ &\leq d(x_1, x_0)(\alpha^{m-1} + \alpha^{m-2} + \cdots + \alpha^n) \\ &= \alpha^n d(x_1, x_0)(\alpha^{m-1-n} + \alpha^{m-2-n} + \cdots + \alpha + 1) \\ &\leq \alpha^n d(x_1, x_0)(1 + \alpha + \alpha^2 + \alpha^3 + \cdots + \alpha^{m-1-n} + \alpha^{m-n} + \cdots) \\ &= \frac{\alpha^n}{1 - \alpha} d(x_1, x_0). \end{aligned}$$

Note that the terms in the bracket in the third line are part of a geometric series, and so by throwing in all terms in that geometric series, the sum is only made bigger. This explains the inequality in the fourth line. To go from the fourth to the fifth line, we use the summation formula for a geometric series, which we can since $\alpha < 1$.
Now suppose we are given $\varepsilon > 0$. We can also assume that $d(x_1, x_0) \neq 0$. SInce if it were zero , then x_0 would be a fixed point, and we would be done. So if $d(x_1, x_0) \neq 0$, then let N such that for any $n \geq N$, $\alpha^n < \varepsilon(1 - \alpha)/d(x_1, x_0)$. We can do this since $\lim_{n\to\infty} \alpha^n = 0$. So if $m, n \geq N$ and $m > n$ then we get that

$$d(x_m, x_n) < \varepsilon,$$

and hence the sequence is Cauchy. Since X is complete, $x_n \to p$ for some $p \in X$.
Claim. $f(p) = p$. That is, p is a fixed point.
Proof. First note that f is Lipshitz and hence continuous. Now consider $g(x) = d(x, f(x))$. Since the distance function is also continuous, g is a continuous function on X. Applying the above estimate to $m = n + 1$, since $x_{n+1} = f(x_n)$,

$$g(x_n) = d(x_n, f(x_n)) = d(x_n, x_{n+1}) \leq \frac{\alpha^n}{1 - \alpha} d(x_1, x_0).$$

Letting $n \to \infty$, sicne $x_n \to p$ and $\alpha^n \to 0$, we see that

$$g(p) = \lim_{n\to\infty} g(x_n) = 0.$$

But $g(p) = 0 \implies d(p, f(p)) = 0$ and hence $p = f(p)$ since distance function is positive definite.

9. Let K be a compact subset of a metric space (X, d) and F a closed subset.

(a) Show that $K \cap F$ is a compact subset.

Solution: Let $\{x_n\}$ be any sequence of points in $K \cap F$. Then since K is compact, there exists a subsequence $\{x_{n_k}\}$ such that $\lim_{k\to\infty} x_{n_k} = p$, where $p \in K$. Now K being compact is also closed, and hence $K \cap F$ is closed. But then since $x_{n_k} \in K \cap F$, p is a limit point of $K \cap F$ and so lies in $K \cap F$. So we have proved that any sequence $\{x_n\}$ in $K \cap F$ has a subsequence converging to a point in $K \cap F$. This proves that $K \cap F$ is compact.

(b) If $K \cap F = \phi$, show that

$$\inf_{x\in K,\ y\in F} d(x,y) > 0.$$

Solution: Let $\alpha = \inf_{x\in K,\ y\in F} d(x,y)$, and let $x_n \in K$ and $y_n \in F$ be a sequence of points such that

$$\lim_{n\to\infty} d(x_n, y_n) = \alpha. \tag{0.2}$$

Since K is compact, there exists a subsequence $x_{n_k} \to p \in K$. Now, suppose $\alpha = 0$. Then it follows from (**??**) that $\lim_{k\to\infty} d(p, y_{n_k}) = 0$, that is $y_{n_k} \to p$. So p is a limit point of F. Since F is closed, this forces $p \in F$. So $p \in K \cap F$ which is a contradiction since $K \cap F = \phi$. And hence $\alpha > 0$.

(c) Providing an example, argue that if K is assumed to be only closed, then the infimum could be zero.

Solution: Let $K = \mathbb{N}$, and let

$$F = \{n + \frac{1}{n} \mid n \in \mathbb{N}\}.$$

Then $K \cap F = \phi$, both K and F are closed, but $\lim_{n\to\infty} d(n, n + \frac{1}{n}) = 0$, and so

$$\inf_{x\in K,\ y\in F} d(x,y) = 0.$$

10. Let (X,d) be a metric space.

(a) If $p_a, \cdots, p_n$ be a finite collection of points, show using only the definition that $X \setminus \{p_1, \cdots, p_n\}$ is an open set.

Solution: Given any $x \in X \setminus \{p_1, \cdots, p_n\}$, let

$$r(x) = \frac{1}{2}\min(d(x,p_1), d(x,p_2), \cdots, d(x,p_n)).$$

Then $B_{r(x)}(x)$ is a ball around x that is contained in $X \setminus \{p_1, \cdots, p_n\}$. This shows that $X \setminus \{p_1, \cdots, p_n\}$ is open.

(b) Give an example of a metric space and a sequence of points $\{p_k\}$ such that $X \setminus \{p_k\}_{k=1}^{\infty}$ is dense, infinite but not open.

Solution: Take $X = \mathbb{R}$, and $\{p_1, p_2, \cdots\}$ to be the set of all rational numbers.

11. Define a sequence of functions on $[0,2]$ by

$$f_n(x) = \sqrt{\frac{x^2+n}{x+n}}.$$

(a) Show that the sequence converges uniformly to some continuous function f on $[0,2]$.

Solution: It is clear that $f_n \to 1$ pointwise on $[0,2]$. We claim that the convergence is uniform. To see this, note that for any $x \in [0,2]$

$$\begin{aligned}
|f_n(x) - 1| &= \left|\sqrt{\frac{x^2+n}{x+n}} - 1\right| \\
&= \frac{|\sqrt{x^2+n} - \sqrt{x+n}|}{\sqrt{x+n}} \\
&= \frac{x|x-1|}{\sqrt{x+n}(\sqrt{x^2+n} + \sqrt{x+n})} \\
&\le \frac{x|x-1|}{2n},
\end{aligned}$$

since $\sqrt{x+n}, \sqrt{x^2+n} \ge \sqrt{n}$. So if $M_n = \sup_{x\in[0,2]} |f_n(x) - 1|$, then

$$M_n \le \frac{1}{n} \to 0.$$

Since $\lim_{n\to\infty} M_n = 0$, it follows that the convergence is uniform.

(b) Compute

$$\lim_{n\to\infty} \int_0^2 f_n(t)\,dt,$$

and justify your answer.

Solution: Each f_n being continuous, is Riemann integrable on $[0,2]$. By the theorem on uniform convergence and integrability,

$$\lim_{n\to\infty} \int_0^2 f_n(t)\,dt = \int_0^2 \lim_{n\to\infty} f_n(t)\,dt = \int_0^2 dt = 2.$$

12. The Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Clearly the function is well defined and finite on $(1,\infty)$.

(a) Show that the series converges uniformly on $[\alpha,\infty)$ for all $\alpha > 1$.

Solution: F any $s \ge \alpha$, $n^{-s} \le n^{-\alpha}$. Since $\alpha > 1$, $\sum n^{-\alpha}$ converges, by the Weierstrass M-test (applied to $M_n = n^{-\alpha}$), the series converges uniformly on $[\alpha,\infty)$.

(b) Show that $\zeta(s)$ is differentiable on $(1,\infty)$ with

$$\zeta'(s) = -\sum_{n=1}^{\infty} \frac{\ln(n)}{n^s}.$$

Solution: Let

$$\zeta_n(s) = \sum_{k=1}^{n} \frac{1}{k^s}.$$

Each ζ_n is differentiable, and by chain rule, since $\frac{d}{ds}n^{-s} = -\ln n n^{-s}$ we have

$$\zeta_n'(s) = -\sum_{k=1}^{n} \frac{\ln k}{k^s}.$$

We want to apply the theorem on uniform convergence and differentiation. For this we have to verify two hypotheses.

- **pointwise convergence of ζ_n.** By part(a), it follows that $\zeta_n(s) \to \zeta(s)$ on $[\alpha, \infty)$ (in fact in part(a) we showed that this convergence is uniform).
- **uniform convergence of derivatives.** Since $\alpha > 1$, there exists an $\varepsilon > 0$ such that $\alpha - \varepsilon > 1$. Now, since
$$\lim_{n\to\infty} \frac{\ln n}{n^\varepsilon} = 0,$$
it follows that there exists a constant C such that
$$\ln n < Cn^\varepsilon$$
for all $n \in \mathbb{N}$. So for $s \in [\alpha, \infty)$,
$$\frac{\ln n}{n^s} \leq C\frac{1}{n^{\alpha-\varepsilon}}.$$
Since $\alpha - \varepsilon > 1$, the series $\sum n^{-\alpha+\varepsilon}$ converges and so by Weierstrass M-test, the series
$$-\sum_{n=1}^{\infty} \frac{\ln n}{n^s}$$
converges uniformly on $[\alpha, \infty)$. Therefore the sequence $\zeta_n'(s)$ (being the partial sums of the above series) converges uniformly on $[\alpha, \infty)$.

Then by the theorem on uniform convergence and differentiation, $\zeta(s)$ being the limit of $\zeta_n(s)$ is differentiable on $[\alpha, \infty)$, and moreover,

$$\zeta'(s) = \lim_{n\to\infty} \zeta_n'(s) = -\sum_{n=1}^{\infty} \frac{\ln n}{n^s}$$