Solutions to Assignment-7

(Due 07/30)

Please hand in all the 8 questions in red

1. Consider the sequence of functions $f_n : [0,1] \to \mathbb{R}$ defined by

$$f_n(x) = \frac{x^2}{x^2 + (1-nx)^2}.$$

(a) Show that the sequence of functions converges pointwise as $n \to \infty$, and compute the limit function $f(x)$.

Solution: For any $x \in [0,1]$,

$$\lim_{n\to\infty} \frac{x^2}{x^2 + (1-nx)^2} = 0.$$

If we let $M_n = \sup_{x\in[0,1]} |f_n(x)|$, then we see that $M_n \geq f_n(1/n) = 1$, and so $\lim_{n\to\infty} M_n \neq 0$. Therefore, f_n does not converge uniformly to 0 on $[0,1]$.

(b) Show that the sequence is not equicontinuous on $[0,1]$.

Solution: Let $x_n = 1/n$ and $y_n = 0$. Then $|x_n - y_n| \to 0$ as $n \to 0$, but

$$|f_n(x_n) - f_n(y_n)| = 1,$$

contradicting the definition of equicontinuity for $\varepsilon = 1$.

(c) Which theorem in the notes implies that f_n does not converge uniformly to f on $[0,1]$?

Solution: We have already shown in part(a) that the convergence is not uniform, but this also follows from part(b) and Theorem 7.2 on page 7 of the notes for Week-7.

(d) Show that $f_n \xrightarrow{u.c} f$ on $[a,1]$ for all $a \in (0,1)$.

Solution: We rewrite

$$f_n(x) = \frac{x^2}{n^2\left(\frac{x^2}{n^2} + \left(x - \frac{1}{n}\right)^2\right)}$$

Let N such that $1/N < a/2$. Then for all $n > N$ and all $x \in [a,1]$, $x - 1/n > a/2$ and so

$$\frac{x^2}{n^2} + \left(x - \frac{1}{n}\right)^2 > \frac{a^2}{4}.$$

Therefore, for $n > N$ and $x \in [a,1]$,

$$|f_n(x)| \leq \frac{4x^2}{n^2a^2} \leq \frac{4}{a^2}n^{-2}.$$

So if $M_n = \sup_{x\in[a,1]} |f_n(x)|$, then

$$0 \le M_n \le \frac{4}{a^2} n^{-2}.$$

By squeeze principle, $\lim_{n\to\infty} M_n = 0$, and so $f_n \xrightarrow{u.c} 0$ on $[a,1]$ for every $a > 0$.

2. Let $\mathcal{F} \subset \mathcal{R}[0,1]$ be the set of all Riemann integrable functions on $[0,1]$ such that $|f(t)| \le M$ for some fixed M. For any $f \in \mathcal{F}$, define $I[f] : [0,1] \to \mathbb{R}$ by

$$I[f](x) = \int_0^{\sqrt{x}} f(t)\, dt.$$

(a) Show that the family $\{I[f] \mid f \in \mathcal{F}\}$ is equicontinuous.

Solution: For any $x, y \in [0,1]$ (say, $y \ge x$), we estimate

$$\begin{aligned} |I[f](y) - I[f](x)| &= \Big| \int_{\sqrt{x}}^{\sqrt{y}} f(t)\, dt \Big| \\ &\le \int_{\sqrt{x}}^{\sqrt{y}} |f(t)|\, dt \\ &\le M(\sqrt{y} - \sqrt{x}). \end{aligned}$$

Now, $g(x) = \sqrt{x}$ is uniformly continuous on $[0,1]$. So given any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x - y| < \delta \implies |\sqrt{y} - \sqrt{x}| < \frac{\varepsilon}{M}.$$

So for this δ,

$$|x - y| < \delta \implies |I[f](y) - I[f](x)| < \varepsilon.$$

Hence the family $\{I[f] \mid f \in \mathcal{F}\}$ is equicontinuous.

Note. If $\mathcal{F}$ was instead restricted to consist of only continuous functions, then the following method can be attempted. We can write $I[f](x) = F(\sqrt{x})$, where F is the anti-derivative

$$F(u) = \int_0^u f(t)\, dt.$$

Since f is continuous, by the fundamental theorem, $F(u)$ is differentiable on $(0,1)$ and continuous on $[0,1]$, with $F'(u) = f(u)$. By chain rule,

$$\frac{d}{dx} I[f](x) = \frac{f(\sqrt{x})}{2\sqrt{x}}.$$

So the derivative is not necessarily bounded. Hence the usual mean value trick will not work in this case.

(b) Show that given any sequence of functions $\{f_n\}$ in $\mathcal{F}$, there exists a sub-sequence $\{f_{n_k}\}$ such that $I[f_{n_k}]$ converges uniformly on $[0,1]$.

Solution: Since $|f(t)| \le M$, we see that $|I[f](x)| \le M$, and hence the family $\{I[f] \mid f \in \mathcal{F}\}$ is uniformly bounded. It is also equicontinuous by part(a). So by Arzela-Ascoli, given any sequence $f_n \in \mathcal{F}$, there exists subsequence f_{n_k} such that $I[f_{n_k}]$ converges uniformly on $[0,1]$.

3. Consider the sequence of functions $f_n : [0,2] \to \mathbb{R}$,

$$f_n(t) = \frac{t^n}{1+t^n},$$

and let $F_n : [0,2] \to \mathbb{R}$ be the anti-derivatives.

(a) Show that $f_n(t)$ converges point-wise on $[0,2]$. What is the limit function?

Solution: The sequence converges pointwise to f where

$$f(t) = \begin{cases} 0, & t \in [0,1) \\ \frac{1}{2}, & t = 1 \\ 1, & t \in (1,2]. \end{cases}$$

(b) Argue, by simply looking at the limit function above, that no subsequence converges uniformly on $[0,2]$.

Solution: Since each f_n is continuous and the limit function is not continuous, by the theorem on uniform convergence and continuity, no subsequence of f_n can converge uniformly to f.

(c) Show that for all $x, y \in [0,2]$,

$$|F_n(x) - F_n(y)| \leq |x - y|.$$

Solution: Since each f_n is continuous, by the fundamental theorem of calculus, F_n is differentiable on $(0,2)$ and continuous on $[0,2]$. Moreover $|F_n'(x) = f_n(x)$ for all $x \in (0,2)$ and hence

$$|F_n'(x)| \leq 1.$$

But then by the mean value theorem, since F_n is continuous on $[0,2]$, for any $x, y \in [0,2]$,

$$|F_n(x) - F_n(y)| = |F_n'(c)||x - y| \leq |x - y|,$$

where c is some number between x and y.

(d) Show that there is a subsequence F_n that converges uniformly on $[0.2]$.

Solution: Since each $|f_n(t)| \leq 1$ on $[0,2]$, by the triangle inequality for integrals, for any $x \in [0,2]$ we have

$$|F_n(x)| \leq \int_0^x |f_n(t)|\, dt \leq x \leq 2.$$

This shows that the sequence $\{F_n\}$ is uniformly bounded on $[0,2]$. On the other hand, by part(c) above, the sequence is also equicontinuous (simply let $\delta = \varepsilon$ in the definition of equicontinuity). So by Arzela-Ascoli, there exists a subsequence that converges uniformly on $[0,2]$.

4. Let $C^0[0,1]$ denote the set of all continuous real valued functions on $[0,1]$. For $f, g \in C^0[0,1]$, define

$$d(f,g) = \sup_{t \in [0,1]} |f(t) - g(t)|.$$

(a) Show that d defines a metric on $C^0[0,1]$.

Solution: We need the following basic fact about supremums (a related property of limsup, was a homework problem) - If f and g are bounded functions on a set E, then

$$\sup_{t\in E}|f(t)+g(t)| \le \sup_{t\in E}|f(t)| + \sup_{t\in E}|g(t)|$$

- $d(f,g)$ **is finite.** Since f and g are continuous and $[0,1]$ is compact, they are bounded. So there exists M such that $|f(t)|,\ |g(t)| < M$ for all $t \in [0,1]$. But then by the above property we see that $d(f,g) < 2M$ and hence is finite.
- **(Positive definiteness)** $d(f,g) >\ge 0$ **and** $d(f,g) = 0$ **if and only if** $f = g$**.** It is clear that $d(f,g) \ge 0$. So suppose $d(f,g) = 0$. Then by definition of d, $|f(t) - g(t)| = 0$ for all $t \in [0,1]$, which shows that $f(t) = g(t)$ for all $t \in [0,1]$.
- **(Symmetry)** $d(f,g) = d(g,f)$**.** Obvious!
- **(Triangle inequality)** $d(f,g) \le d(f,h) + d(g,h)$**.** By the usual triangle inequality for $|\cdot|$, for any $t \in [0,1]$,

  $$|f(t) - g(t)| \le |f(t) - h(t)| + |g(t) - h(t)|.$$

  Taking supremum on both sides and using the property above, we prove the required triangle ienquality.

(b) Show that $f_n \to f$ in this metric, if and only if $f_n \to f$ uniformly on $[0,1]$.

Solution: Note that $f_n \xrightarrow{d} f$, if and only if for any $\varepsilon > 0$, there exists an N such that

$$d(f_n, f) < \varepsilon$$

whenever $n > N$. But by definition of the distance function this is equivalent to the statement that for all $\varepsilon > 0$, there exists an $N > 0$ such that for all $n > N$,

$$\sup_{t\in[0,1]} |f(t) - f_n(t)| < \varepsilon,$$

which in turn is equivalent to the statement that $f_n \xrightarrow{\text{u.c}} f$ on $[0,1]$.

(c) Show that $(C^0[0,1], d)$ is a complete metric space, that is every Cauchy sequence is convergent. **Note.** A sequence $\{x_n\}$ in a metric space (X,d) is said to be Cauchy $\forall\varepsilon > 0$, there exists N such that for all $n, m > N$, $d(x_n, x_m) < \varepsilon$. We will talk about completeness in more detail in class on Monday, but this is enough to solve the problem.

Solution: Let $\{f_n\}$ be a Cauchy sequence in $(C^0[0,1], d)$. That is for any $\varepsilon > 0$, there exists N such that for any $n, m > N$ we have

$$d(f_n, f_m) < \varepsilon.$$

By the definition of the distance for any $t \in [0,1]$,

$$|f_n(t) - f_m(t)| < \varepsilon.$$

Hence the sequence $\{f_n\}$ is uniformly Cauchy, and so $f_n \to f$ uniformly on $[0,1]$ for some function f. Moreover, since f_n is continuous for each n, the limit f will also be continuous and hence $f \in C^0[0,1]$. But then by part(b) above, we have that $f_n \xrightarrow{d} f$, and this shows that any Cauchy sequence has a limit in $C^0[0,1]$, proviung that the space is complete.

5. Let (X,d) be a metric space. The boundary ∂E and frontier dE of a set $E \subset X$ are defined respectively as

$$\partial E = \overline{E} \setminus \text{int}(E),$$
$$dE = \overline{E} \setminus E.$$

where $\overline{E}$ is the closure of the set E and $\text{int}(E)$ is the interior. Consider the following subset of $\mathbb{R}^2$,

$$E = \{(x,y) \mid 0 < x^2 + y^2 < 1\} \cup \{(x,0) \mid 1 \le x \le 2\}.$$

(a) Draw a **neat and labelled** diagram in the x-y plane indicating the subset E. Open sets can be shown with dotted lines.

(b) Write down the sets $\overline{E}$, $\text{int}(E)$, ∂E and dE.

6. If A and B denote arbitrary subsets of a metric space (X,d), prove the following properties. Here $\text{int}(A)$ denotes the interior of A and $\overline{A}$ as usual denotes the closure.

(a) $\text{int}(A) = X - \overline{X - A}$.

Solution: Since $A^c \subset \overline{A^c}$, clearly $X - \overline{X-A} \subset A$. Moreover $X - \overline{X-A}$ is open and so $X - \overline{X-A} \subseteq \text{int}(A)$ since $\text{int}(A)$ is the largest open set contained in A. To complete the proof, we show $\text{int}(A) \subseteq X - \overline{X-A}$. So let $p \in \text{int}(A)$. Then there is some $r > 0$ such that $B_r(p) \subset A$. So p cannot be a limit point for $X - A$, and hence cannot belong to $\overline{X-A}$, and hence lies in $X - \overline{X-A}$, completing the proof.

(b) If $\text{int}(A) = \text{int}(B) = \phi$, and A is closed, then $\text{int}(A \cup B) = \phi$. If A is not necessarily closed, given an example where $\text{int}(A \cup B) = X$.

Solution: Suppose $\text{int}(A \cup B)$ is non-empty, and let p be a point in the interior. Then there is some $r > 0$ such that $B_r(p) \subset A \cup B$. If $p \notin A$, then $p \in B$, and moreover, since A is closed, p cannot be a limit point of A. So there is some radius r' such that $B_{r'}(p)$ does not intersect A. But then if $\varepsilon = \min(r, r')$, then $B_\varepsilon(p) \subset A \cup B$ but $B_\varepsilon(p) \cap A = \phi$. So $B_\varepsilon(r) \subset B$, making p an interior point of B, a contradiction. Now, suppose $p \in A$. Since int(A) is empty, there is some $q \in B_r(p)$ which is not in A. But A^c is open and so there is some r' such that for any $\varepsilon < r'$, $B_\varepsilon(q) \subset A^c$. Choose ε small enough (say smaller $r - d(p,q)$) such that $B_\varepsilon(q) \subset B_r(p) \subset A \cup B$. But then $B_\varepsilon(q)$ will have to be contained in B making q and interior point of B which is a contradiction.

To show that one of the sets has to be closed. Consider $A = \mathbb{Q}$ and $B = \mathbb{R} \setminus \mathbb{Q}$. Then both A and B have empty interiors, but their union is all of $\mathbb{R}$.

(c) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$. Given an example of strict inclusion.

Solution: The inclusion follows from 2(a) above, since any limit point of $A \cap B$ is also a limit point of BOTH A and B. For strict inclusion, consider $A = (-1, 0)$ and $B = (0, 1)$. Then $0 \in \overline{A} \cap \overline{B}$, but $A \cap B$ is empty and so has an empty closure.

7. Given $A \subset (X,d)$, let $L(A)$ be the set of limit points of A.

(a) Show that $L(A)$ is closed.

Solution: If not, then there exists a point $p \in \overline{L(A)} \setminus L(A)$. Then there is a sequence of points $x_n \in L(A)$ such that $x_n \to p$. For each n, since $x_n \in L(A)$, $B_{1/n}(x_n) \cap A$ is non empty. So let $y_n \in B_{1/n} \cap A$. Then we claim that $y_n \to p$. But then since $y_n \in A$, this proves that p is a limit point of A, which is a contradiction. To prove the claim, let $\varepsilon > 0$. Then there exists N_1 such that $d(x_n, p) < \varepsilon/2$ whenever $n > N_1$. Also, let N_2 such that $1/N_2 < \varepsilon/2$. Letting $N = \max(N_1, N_2)$ we see that for $n > N$,

$$d(y_n, p) \le d(y_n, x_n) + d(x_n, p) < \frac{1}{n} + \frac{\varepsilon}{2} < \varepsilon.$$

(b) Show that if p is a limit point of $A \cup L(A)$, then p is also a limit point of A. Is it necessarily a limit point of $L(A)$?

Solution: Let p be a limit point of $A \cup L(A)$, but not of A. That is, $p \notin L(A)$. Then there exists an $r_1 > 0$ such that $B_r(p) \cap A$ is either empty or consists of only p. Moreover, since $L(A)$ is closed, there exists $r_2 > 0$ such that $B_r(p) \cap L(A) = \phi$. So if $r = min(r_1, r_2) > 0$, $B_r(p) \cap (A \cup L(A))$ is either empty or consists of only p. So p cannot be a limit point of $A \cup L(A)$. Contradiction!

For the second part, the answer is NO. For instance, let $A = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$. Then $L(A) = \{0\}$, and 0 is also a limit point of $A \cup L(A)$. But 0 is an isolated point of $L(A)$.

8. Let (X, d_X) and (Y, d_Y) be metric spaces. Suppose $f : X \to Y$ is a continuous

(a) For any $y \in Y$, show that $Z_y = \{x \in X \mid f(x) = y\}$ is a closed set.

Solution: The singleton set $\{y\}$ is closed in (Y, d_Y). By the characterization of continuity in terms of inverse images of closed sets (Theorem 8.11(c) in Week-7 notes), it follows that $Z_y = f^{-1}(\{y\})$ is closed.

(b) Suppose now $Y = \mathbb{R}$ with the standard Euclidean metric $|\cdot|$. If for some $p \in X$, $f(p) > 0$, then show that there is some $\delta > 0$ such that for all $x \in B_\delta(p)$, $f(x) > 0$.

Solution: Let $\varepsilon = f(p)/2 > 0$. Then since f is continuous, there exists a $\delta > 0$ such that for all $x \in B_\delta(p)$,

$$|f(x) - f(p)| < \varepsilon.$$

But then since $f(p) = 2\varepsilon$, this shows that for all $x \in B_\delta(p)$,

$$f(x) > \varepsilon > 0.$$

9. Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$ be a continuous function. Prove that

$$f(\overline{E}) \subset \overline{f(E)}$$

for any subset $E \subset X$. Show by example that the inclusion can be strict.

Solution: Let $q \in f(\overline{E})$. That is $q = f(p)$ for some $p \in \overline{E}$. If $p \in E$, then $q \in f(E)$ and so there is nothing to prove. So suppose $p \in \overline{E} \setminus E$. Then there exists $x_n \in E$ such that $x_n \to p$. Since f is continuous $f(x_n) \to f(p)$. But then $q = f(p) \in \overline{f(E)}$, since $f(x_n) \in f(E)$. Done! Consider the function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \frac{1}{1+x^2},$$

and let $E = (0, \infty)$. Then $f(\overline{E}) = (0, 1]$, while $\overline{f(E)} = [0, 1]$.

10. Show, using only the definition of compactness, that the set

$$K = \{\frac{1}{n} \mid n \in \mathbb{N}\}$$

is NOT compact, while the set $K \cup \{0\}$ is compact.

Solution: Consider the sequence $x_n = 1/n$ in K. Then it converges to $0 \notin K$. So any subsequence also converges to 0. That is, there is no subsequence which converges to a number in K, and hence K cannot be compact. On the other hand, consider $E = K \cup \{0\}$. Let $\{x_n\}$ be any sequence in E. If the sequence is finite, then there is of course a subsequence that is constant, and hence convergent. So suppose the sequence is infinite. Let n_1 such that $s_{n_1} \neq 0$. Then let $n_2 > n_1$ such that $x_{n_1} < x_{n_2}$ and $x_{n_2} \neq 0$. Having chosen $n_1 < n_2 < \cdots < n_{k-1}$, let $n_k > n_{k-1}$ such that $x_{n_k} < x_{n_{k-1}}$ and $x_{n_k} \neq 0$. We claim that $\{x_{_k}\}$ converges to zero. To see this, let $\varepsilon > 0$ and N such that $1/N < \varepsilon$. Since there are N elements in K greater than or equal to $1/N$, it follows that there exists some m such that for all $k > m$,

$$0 < x_{n_k} < \frac{1}{N} < \varepsilon.$$

This shows that $\lim_{k\to\infty} x_{n_k} = 0$. So given any sequence $\{x_n\}$ in K we have managed to extract a subsequence $\{x_{n_k}\}$ which converges to an element in K, namely, to zero.

11. Let $\mathbb{Q}$ be the set of rationals with the usual distance function $d(r, s) = |r - s|$. Let E be the set of rationals r satisfying $2 < r^2 < 3$. Show that E is closed and bounded but not compact. This shows that the Hein-Borel or Bolzano-Weierstrass theorem is not true in a general metric space.

Solution: Let us denote the induced metric on $\mathbb{Q}$ be $d_{\mathbb{Q}}$, and let us denote balls in this metric by $B_r^{\mathbb{Q}}(p)$. That is,

$$B_r^{\mathbb{Q}}(p) = \{q \in \mathbb{Q} \mid |p - q| < r\}.$$

Now it is easy to see that $3/2 \in E$ and that $E \subset B_1^{\mathbb{Q}}(3/2)$, and so E is bounded. For closedness, suppose $q \in \mathbb{Q}$ is a limit point of E and suppose $q \notin E$. Then either $q^2 < 2$ or $q^2 > 3$ since q^2 cannot be equal to 2 or 3. Suppose $q^2 < 2$. Then $q < \sqrt{2}$. Choose $\varepsilon > 0$ such that $q + \varepsilon < \sqrt{2}$. But then $B_\varepsilon^{\mathbb{Q}}(q) \cap E = \phi$, and so q could not have been a limit point. Contradiction! The other case is similar. This shows that E is closed in $\mathbb{Q}$. To show that the set is not compact, consider the decimal approximations to $\sqrt{3}$. That is $d_0 = 1$, $d_1 = 7$, and having chosen $d_0, d_1, \cdots, d_{n-1}$, we let d_n be the largest natural number such that

$$d_0 + \frac{d_1}{10} + \cdots + \frac{d_n}{10^n} < \sqrt{3}.$$

Then the sequence $a_1 = d_0.d_1$, $a_2 = d_0.d_1d_2, \cdots, a_n = d_0.d_1d_2 \cdots d_n, \cdots$ lies in E but no subsequence converges to a rational number (in fact every subsequence converges to $\sqrt{3}$, which is of course not rational).

12. Recall that $C^0[0,1]$ denotes the set of continuous functions on $[0,1]$. We endow it with the usual metric

$$d(f,g) = \sup_{t\in[0,1]} |f(t) - g(t)|.$$

Define a function $T : C^0[0,1] \to C^0[0,1]$ by

$$T[f](x) = \int_0^x f(t)\,dt.$$

Let $\mathcal{K} \subset C^0[0,1]$ be a bounded set.

(a) Show that T is a continuous function. Is it injective? **Hint.** To show continuity, it is enough to show (Why?) that if $f_n \xrightarrow{u.c} f$, then $T[f_n] \xrightarrow{u.c} T[f]$.

Solution: For $f, g \in C^0[0,1]$, and any $t \in [0,1]$, by definition,

$$|f(t) - g(t)| \le d(f,g).$$

So for any $x \in [0,1]$,

$$\begin{aligned} |T[f](x) - T[g](x)| &= \Big| \int_0^x [f(t) - g(t)]\,dt \Big| \\ &\le \int_0^x |f(t) - g(t)|\,dt \\ &\le d(f,g) \int_0^x \,dt \\ &\le d(f,g). \end{aligned}$$

Taking supremum over all $x \in [0,1]$, we see that

$$d(T[f], T[g]) \le d(f,g).$$

So given an $\varepsilon > 0$, let $\delta = \varepsilon$. Then

$$d(f,g) < \delta \implies d(T[f], T[g]) < \varepsilon.$$

This shows that T is a continuous map. We claim that the function is injective. It is enough to show that if $T[f](x) = 0$ for all $x \in [0,1]$, then $f(t) = 0$ for all $t \in [0,1]$. To see this, note that if $T[f](x) = 0$ for all x, then for any $x, y \in [0,1]$, $x < y$,

$$\int_x^y f(t)\,dt = 0.$$

Now, suppose $f(t_0) > 0$ at some point $t_0 \in [0,1]$. Since f is continuous, there exists $\delta > 0$ such that for any $t \in (t_0 - \delta, t_0 + \delta)$, $f(t) > 0$. But then

$$\int_{t_0-\delta}^{t_0+\delta} f(t)\,dt > 0,$$

which is a contradiction. So there is not point where f is strictly positive. Similarly there is no point where f is strictly negative. Hence f has to be identically zero.

(b) Show that the set $\overline{T(\mathcal{K})}$ is a compact subset of $C^0[0,1]$. **Hint.** Use the Version-2 of Ascoli-Arzela.

Solution: Since $\mathcal{K}$ is a bounded set, there exists an M such that $d(f, 0) < M$ for all $f \in K$. Here 0 denotes the zero function, that is the function that vanished on all of $[0, 1]$. By the definition of d, this means that

$$|f(t)| < M, \ \forall t \in [0,1], \ \forall f \in \mathcal{K}.$$

- **Closed.** Trivially since it is the closure of a set.
- **Bounded.** For any $f \in \mathcal{K}$, since $|f(t)| < M$ for all $t \in [0, 1]$, we see that

$$\begin{aligned} |T[f](x)| &= \Big| \int_0^x f(t)\, dt \Big| \\ &\le \int_0^x |f(t)|\, dt \\ &< M, \end{aligned}$$

  and so

$$d(T[f], 0) < M.$$

  This shows that $T(\mathcal{K})$ is bounded, and hence $\overline{T(\mathcal{K})}$ is also bounded.
- **Equicontinuous.** Let $\varepsilon > 0$, and $\delta = \delta(\varepsilon)$ to be picked later. For any $f \in C^0[0, 1]$, by the fundamental theorem of calculus, $T[f]$ is differentiable on $[0, 1]$, and moreover,

$$T[f]'(x) = f(x).$$

  So if $f \in \mathcal{K}$, we have that

$$|T[f]'(x)| < M$$

  for all $x \in [0, 1]$. Then by the mean value theorem, for any $x, y \in [0, 1]$,

$$|T[f](x) - T[f](y)| < M|x - y|.$$

  Another way to see this is to directly estimate the integral. That is if $x, y \in [0, 1]$ with say $x > y$, then

$$\begin{aligned} |T[f](x) - T[f](y)| &= \Big| \int_y^x f(t)\, dt \Big| \\ &\le \int_y^x |f(t)|\, dt \\ &< M|x - y|. \end{aligned}$$

  In any case, if we take $\delta = \varepsilon/M$, then

$$|x - y| < \delta \implies |T[f](x) - T[f](y)| < \varepsilon.$$

  This shows that $T(\mathcal{K})$ is equicontinuous. To see that the closure is also equicontinuous, we use the $\varepsilon/3$ trick. So let $\varepsilon > 0$, and $\delta = \varepsilon/3M$. Then by the abvove argument for any $f \in \mathcal{K}$,

$$|x - y| < \delta \implies |T[f](x) - T[f](y)| < \frac{\varepsilon}{3}.$$

  Now if $g \in \overline{T(\mathcal{K})}$, there exists an $f \in \mathcal{K}$ such that $d(T[f], g) < \varepsilon/3$. That is for any $x \in [0, 1]$,

$$|T[f](x) - g(x)| < \frac{\varepsilon}{3}.$$

Then if $|x-y|<\delta$,

$$\begin{aligned}|g(x)-g(y)| &\leq |g(x)-T[f](x)| + |T[f(x)-T[f](y)| + |T[f](y)-g(y)| \\ &\leq \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.\end{aligned}$$

So given an $\varepsilon>0$, there exists a $\delta=\delta(\varepsilon)$ (in this case $\delta=\varepsilon/3M$ works) such that for any $g\in\overline{T(\mathcal{K})}$,

$$|x-y|<\delta \implies |g(x)-g(y)|<\varepsilon,$$

and hence $\overline{T(\mathcal{K})}$ is equicontinuous.

Then by the version-2 of Arzela-Ascoli, the set $\overline{T(\mathcal{K})}$ is compact.

13. This exercise shows that even in a complete metric, a closed and bounded set need not be compact. Let

$$l^\infty(\mathbb{R});=\{\{a_k\}_{k=1}^\infty \mid a_k\in\mathbb{R}, \text{ and } \sup_k a_k<\infty\}.$$

That is, $l^\infty(\mathbb{R})$ is the set of all *bounded* sequences of real numbers. Note that the M will vary from sequence to sequence. For two sequences $A=\{a_n\}$ and $B=\{b_n\}$, define

$$d(A,B)=\sup_k |a_k-b_k|.$$

(a) For any two sequences $A,B\in l^\infty(\mathbb{R})$, show that $d(A,B)$ is a finite number.

Solution: Suppose $A=\{a_n\}$ and $B=\{b_n\}$. Let $\alpha=\sup|a_n|$ and $\beta=\sup|b_n|$, which are finite since $A,B\in l^\infty(\mathbb{R})$. Then for any n,

$$|a_n-b_n|\leq|a_n|+|b_n|\leq\alpha+\beta.$$

Hence the sup is finite and $d(A,B)$ is finite.

(b) Show that d is a metric on $l^\infty(\mathbb{R})$.

Solution: The symmetry axiom is trivial. Suppose $A=\{a_k\}$ and $B=\{b_k\}$. THen clearly $d(A,B)\geq 0$. Also if $d(A,B)=0$, then clearly $a_k=b_k$ for all k and so $A=B$. This shows that d is positive definite. For the triangle inequality, note that if $A=\{a_k\}$, $B=\{b_k\}$ and $C=\{c_k\}$ are three elements, for each fixed k,

$$|a_k-c_k|\leq|a_k-b_k|+|b_k-c_k|.$$

Taking sup it follows that $d(A,C)\leq d(A,B)+d(B,C)$ which verifies the triangle inequality. Next we show completeness.

(c) Let E_n be the sequence with 1 at the n^{th} place and zero everywhere else, and let O be the sequence with zeroes everywhere. What is $d(E_n,O)$? $d(E_n,E_m)$ for $n\neq m$?

Solution: From the definition, $d(E_n,O)=1$ and $d(E_n,E_m)=1$ whenever $n\neq m$.

(d) Show that the set $\overline{B_1(O)}$ is closed and bounded, but not compact. **Hint.** Show that the sequence E_n from above has no limit point.

Solution: Consider the sequence $\{E_n\}$from above. Then $d(E_n, 0) = 1$, the sequence is contained in $\overline{B_1(O)}$. Next, since $d(E_n, E_m) = 1$ for all $n \neq m$, clearly the sequence E_n is not Cauchy and cannot have a limit point. But every infinite sequence in a compact metric space has a limit point, and so $\overline{B_1(O)}$ cannot be compact.

An application of Arzela-Ascoli to differential equations

The problems in this section are only for the purpose of entertainment, and will not have any bearing whatsoever on your performance in this course.

Our aim (following Rudin, exercise 7.25) is to show that there exists a function $u : [0,1] \to \mathbb{R}$, continuous on $[0,1]$ and differentiable on $(0,1)$ solving the following initial value problem (IVP)

$$\begin{cases} u'(t) = \sin(u(t)), \\ u(0) = c. \end{cases}$$

For a fixed n, and $i = 0, 1, \cdots, n$, put $t_i = i/n$, and let $u_n : [0,1] \to \mathbb{R}$ be the continuous function defined by $u_n(0) = c$ and such that

$$u_n'(t) = \sin(u_n(t_i)), \ t_i < t < t_{i+1}.$$

You should think of u_n as the n^{th} approximation solution to the equation. Essentially, starting at x_0, between x_i and x_{i+1}, the graph of u_n consists of straight line segments with slopes given by $\sin(u_n(x_i))$ (graph the first few functions, say u_1 and u_2). Note that u_n is differentiable everywhere except at $t = t_i$.

Next, define

$$\Delta_n(t) = \begin{cases} u_n'(t) - \sin(u_n(t)), \ t \neq t_i \\ 0, \ \text{otherwise}. \end{cases}$$

So Δ_n measures how far our approximate solutions are from being actual solutions. Moreover, by the definition of Δ_n,

$$u_n(t) = c + \int_0^t [\sin(u_n(t)) + \Delta_n(t)]\, dt.$$

1. Show that on $[0,1]$, $|u_n'(t)| \leq 1$ (wherever it exists), $|\Delta_n(t)| \leq 2$, $\Delta_n(t) \in \mathcal{R}[0,1]$, and $|u_n(t)| \leq |c| + 1$.

2. $\{u_n\}$ is equicontinuous on $[0,1]$. **Note.** You cannot directly apply mean value theorem, since u_n is not differentiable everywhere on $[0,1]$.

3. From this deduce that there exists a subsequence, say $\{u_{n_k}\}$ which converges uniformly to some u on $[0,1]$.

4. Prove that $\sin(u_{n_k}(t)) \xrightarrow{u.c} \sin(u(t))$ on $[0,1]$.

5. From this deduce that $\Delta_{n_k}(t) \xrightarrow{u.c} 0$ on $[0,1]$, since

$$\Delta_n(t) = \sin(u_n(t_i)) - \sin(u_n(t))$$

   on (t_i, t_{i+1}). **Note.** You have to show that the entire sequence $\Delta_n(t)$ converges uniformly to zero, not just $\Delta_{n_k}(t)$.

6. Hence, show that

$$u(t) = c + \int_0^t \sin(u(t))\, dt.$$

   From this, conclude that $u(t)$ solves the initial value problem. Why will this argument not work, if you can only establish pointwise convergence of $\{u_{n_k}\}$?