Assignment-6

(Due 07/30)

1. Let sequences f_n and g_n converge uniformly on some set $E \subset \mathbb{R}$ to f and g respectively

   (a) Construct an example such that $f_n g_n$ does not converge uniformly on E.

   Solution: Take $f_n = g_n = x + 1/n$ and $E = \mathbb{R}$. Clearly $f_n, g_n \to x$ uniformly on $\mathbb{R}$. Now $f_n g_n = (x + 1/n)^2$, and we claim that this does not converge uniformly to x^2. To see this, we let $h_n(x)$ be the sequence of the squares and expand

   $$h_n(x) = \left(x + \frac{1}{n}\right)^2 = x^2 + \frac{2x}{n} + \frac{x^2}{n^2}.$$

   But then we see that

   $$|h_n(n) - x^2| = 2 + \frac{1}{n} \geq 2,$$

   no matter how big of an n we choose. This contradicts uniform convergence.

   (b) Prove that $f_n g_n$ does converge uniformly if f and g are bounded on E.

   Solution: Let $|f(t)|, g(t)| < M$ for all $t \in E$. Without loss of generality, we can assume that $M > 1$. Given $\varepsilon > 0$, choose N such that for all $t \in E$ and all $n > N$,

   $$|f(t) - f_n(t)|,\ |g(t) - g_n(t)| < \frac{\varepsilon}{3M}.$$

   Note also that if ε is small enough, then for all $n > N$ and all $t \in E$,

   $$|f_n(t)| \leq |f_n(t) - f(t)| + |f(t)| \leq \frac{\varepsilon}{3M} + M < M + 1.$$

   Next, note that

   $$f_n(t)g_n(t) - f(t)g(t) = f_n(t)g_n(t) - f_n(t)g(t) + f_n(t)g_(t) - f(t)g(t),$$

   and so by triangle inequality, for all $t \in E$ and all $n > N$,

   $$\begin{aligned} |f_n(t)g_n(t) - f(t)g(t)| &\leq |f_n(t)g_n(t) - f_n(t)g(t)| + |f_n(t)g_(t) - f(t)g(t)| \\ &= |f_n(t)||g_n(t) - g(t)| + |g(t)||f_n(t) - f(t)| \\ &\leq (M+1)\frac{\varepsilon}{3M} + M\frac{\varepsilon}{3M} < \varepsilon. \end{aligned}$$

2. Show that the sequence of functions

   $$f_n(x) = \frac{nx}{n+1}$$

   does not converge uniformly on all of $\mathbb{R}$, but does converge uniformly on bounded intervals (a, b). What is the point-wise limit on $\mathbb{R}$?

Solution: We show that $f_n(x) \to x$ point-wise on $\mathbb{R}$.

$$|f_n(x) - x| = \left|\frac{nx}{n+1} - x\right| = \frac{|x|}{n+1}.$$

At $x = 0$, $f_n(0) = 0$ for all n, and so there is nothing to prove. If $x \neq 0$, given any $\varepsilon > 0$, we can pick N large enough so that $|x|/(N+1) < \varepsilon$. THen for $n > N$ we see that

$$|f_n(x) - x| < \varepsilon.$$

Of course the N depends on x and hence we have only proved point-wise. But notice that if $x \in (a, b)$ in a bounded interval, then there is an M such that $|x| < M$.Then we can simply choose $N > M/\varepsilon$, then for any $n > N$ and $x \in (a, b)$, it is easy to see that

$$|f_n(x) - x| < \varepsilon,$$

and hence the convergence is uniform on bounded intervals.

We now claim that the convergence is NOT uniform on all of $\mathbb{R}$.

That is we need to show that there exists $\varepsilon > 0$, a subsequence $n_k \to \infty$ and points $x_{n_k} \in \mathbb{R}$ such that

$$|f_{n_k}(x_{n_k}) - x_{n_k}| > \varepsilon.$$

For this, let $\varepsilon = 1$, $n_k = k$ and $x_k = k + 2$. Then by the calculation above

$$|f_k(x_k) - x_k| = \frac{|x_k|}{k+1} = \frac{k+2}{k+1} > 1.$$

Note that we cannot use the method of M_n, since f_n is not a bounded function on $\mathbb{R}$.

3. If $g : [0, 1] \to \mathbb{R}$ is a continuous function such that $g(1) = 0$, show that the sequence of functions $\{g(x)x^n\}_{n=1}^{\infty}$ converges uniformly on $[0, 1]$.

Solution: We claim that the sequence converges uniformly to zero. Let $\varepsilon > 0$. Since $g(1) = 0$ and g is continuous at 1, there exists a δ such that

$$|g(x)| < \varepsilon$$

for all $x \in [1 - \delta, 1]$. So for $x \in [1 - \delta.1]$,

$$|g(x)x^n| < |g(x)| < \varepsilon.$$

Since g is also bounded on $[0, 1]$, letting $M = \sup_{t \in [0,1]} |g(t)|$, for $x \in [0, 1 - \delta]$ we have

$$|g(x)x^n| \leq M|x|^n \leq M(1 - \delta)^n.$$

Choose N big enough so that $(1 - \delta)^n < \varepsilon/M$ for all $n > N$. Then for all $n > N$ and all $x \in [0, 1]$,

$$|g(x)x^n| < \varepsilon,$$

showing that

$$g(x)x^n \xrightarrow{\text{u.c}} 0$$

on $[0, 1]$.

4. (a) Define a sequence of functions by

$$f_n(x) = \begin{cases} 1, & x = 1, \frac{1}{2}, \cdots, \frac{1}{n} \\ 0, & \text{otherwise.} \end{cases}$$

Calculate the pointwise limit function f. Is each f_n continuous at zero? Does $f_n \to f$ uniformly on $\mathbb{R}$. Is f continuous at 0?

Solution: The limit function is

$$f(x) = \begin{cases} 1, & x = 1, \frac{1}{2}, \frac{1}{3} \cdots \\ 0, & \text{otherwise.} \end{cases}$$

The function f_n is continuous everywhere except at $x = 1, 1/2, \cdots, 1/n$, whereas the limit function is discontinuous at the reciprocals of all natural numbers. The theorem on uniform convergence and continuity does not automatically imply that the convergence is not uniform, but we nevertheless claim that f_n does not converge uniformly. Suppose the convergence is uniform, then there exists an N such that for all $x \in \mathbb{R}$,

$$|f_N(x) - f(x)| \leq \frac{1}{2}.$$

But then if $x = \frac{1}{N+1}$, we have that

$$\left| f_N\left(\frac{1}{N+1}\right) - f\left(\frac{1}{N+1}\right) \right| = 1,$$

a contradiction.

(b) Repeat the exercise with the functions

$$g_n(x) = \begin{cases} x, & x = 1, \frac{1}{2}, \cdots, \frac{1}{n} \\ 0, & \text{otherwise,} \end{cases}$$

and

$$h_n(x) = \begin{cases} x, & x = 1, \frac{1}{2}, \cdots, \frac{1}{n-1} \\ 1, & x = \frac{1}{n} \\ 0, & \text{otherwise.} \end{cases}$$

Solution:

- **The sequence $g_n(x)$.** The pointwise limit is clearly

$$g(x) = \begin{cases} x, & x = 1, \frac{1}{2}, \cdots \\ 0, \text{otherwise} \end{cases}$$

Again, we note that g_n is continuous everywhere except at $x = 1, 1/2, \cdots, 1/n$ while $g(x)$ is continuous everywhere except the reciprocals of natural numbers.

Claim. $g_n \xrightarrow{u.c} g$ on $\mathbb{R}$.

Proof. Let us consider $M_n = \sup_{\mathbb{R}} |g_n(x) - g(x)|$. Now, $|g_n(x) - g(x)| = 0$ everywhere, except when $x = 1/m$ for $m > n$, in which case the difference is $1/m$. So

$$M_n = \frac{1}{n+1} \xrightarrow{n \to \infty} 0,$$

and hence $g_n \xrightarrow{u.c} g$ on $\mathbb{R}$. $\square$

- **The sequence $h_n(x)$.** The pointwise limit in this case is

$$h(x) = \begin{cases} x, & x = 1, \frac{1}{2}, \cdots \\ 0, \text{otherwise}, \end{cases}$$

exactly as above. Again h_n is continuous everywhere except at $x = 1, 1/2, \cdots, 1/n$ while $h(x)$ is continuous everywhere except the reciprocals of natural numbers. But in this case we claim.

Claim. h_n does not converge uniformly to h on $\mathbb{R}$.

Proof. To see this, we again consider $M_n = \sup_{\mathbb{R}} |h_n(x) - h(x)|$. Now, $|g_n(x) - g(x)| = 0$ everywhere, except when $x = 1/m$ for $m \geq n$. Moreover, we have that

$$\Big| h_n\Big(\frac{1}{m}\Big) - h\Big(\frac{1}{m}\Big)| = \begin{cases} \frac{1}{m}, & m > n \\ 1 - \frac{1}{n}, & m = n, \end{cases}$$

and so $M_n = 1 - 1/n$ for $n > 2$ and hence

$$\lim_{n\to\infty} M_n = 1 \neq 0.$$

$\square$.

5. Let $\{f_n\}$ be a sequence of real-valued continuous functions $f_n : E \to \mathbb{R}$ for some subset $E \subset \mathbb{R}$. Suppose $f_n \xrightarrow{u.c} f$ on E. Show that
$$f_n(x_n) \to f(x)$$
for every sequence of points $x_n \to x$ in E. Is the conclusion true if $f_n \to f$ only pointwise? Either provide a proof, or a counter example. Is the converse of the above statement true?

Solution: Given $\varepsilon > 0$, there exists N_1 such that for any $n > N_1$ and any $y \in E$,

$$|f(y) - f_n(y)| < \frac{\varepsilon}{2}.$$

Since f is continuous, there also exists N_2 such that for all $n > N_2$,

$$|f(x_n) - f(x)| < \frac{\varepsilon}{2}.$$

Now let $N = \max(N_1, N_2)$. If $n > N$ we then have

$$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

The conclusion does not hold assuming only pointwise convergence. Consider for instant $f_n(x) = x^n$ on $[0, 1]$. Then $f_n \to f$, pointwise, where

$$f(x) = \begin{cases} 1, & x = \text{‘} \\ , 0, & \text{otherwise}. \end{cases}$$

Now let $x_n = 1 - 1/n \to 1$. Then

$$f_n(x_n) \to e^{-1} \neq f(1) = 1.$$

The converse is also not true. Consider the same function as above, $f_n(x) = x^n$, but on $(0,1)$. Then $f_n \to f = 0$ pointwise but not uniformly. For any $x \in (0,1)$ and any sequence $x_n \to x$, we claim that $f_n(x_n) \to 0 = f(x)$. To see this, let $x < \delta < 1$. Then since $x_n \to x$, for N large enough, $x_n < \delta$. but then $f_n(x_n) < \delta^n \xrightarrow{n\to\infty} 0$. This proves the claim. But since f_n does not converge uniformly to f, we see that the converse does not hold.

6. For $n = 1, 2, \cdots$ and $x \in \mathbb{R}$, define

$$f_n(x) = \frac{x}{1 + nx^2}.$$

Show that $\{f_n\}$ converges uniformly to a differentiable function f on $\mathbb{R}$, and that the equation

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

is correct for all $x \neq 0$ but false at $x = 0$. Why does this not contradict the theorem on uniform convergence and differentiation?

Solution: It is clear that the pointwise limit is 0. Now by completing squares it is easy to see that $1 + nx^2 > 2x\sqrt{n}$, and so

$$|f_n(x)| = \frac{x}{1 + nx^2} < \frac{1}{2\sqrt{n}}.$$

Given $\varepsilon > 0$ choosing $N > 2\varepsilon^2$, we see that the sequence converges to zero uniformly on $\mathbb{R}$. Using quotient rule,

$$f_n'(x) = \frac{(1 + nx^2) - 2nx^2}{(1 + nx^2)^2} = \frac{1 - nx^2}{(1 + nx^2)^2}.$$

It is easy to see (by pulling out an n^2 from the denominator and n from the numerator, or by using L'Hospital's rule) that if $x \neq 0$ then $f_n'(x) \to 0 = f'(x)$. If $x = 0$, then $f'(0) = 0$ but $f_n'(0) = 1$ for all n, and so

$$\lim_{n\to\infty} f_n'(0) \neq f'(0).$$

7. Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of bounded functions. That is for each n, there exists an M_n such that

$$|f_n(t)| < M_n$$

for all $t \in [0,1]$. Suppose in addition that $f_n \xrightarrow{u.c} f$.

(a) Show that f is bounded on $[0,1]$. **Hint.** Apply the definition of uniform convergence with $\varepsilon = 1$.

Solution: There exists an N such that

$$|f(t) - f_N(t)| < 1$$

for all $t \in [0,1]$. But then by triangle inequality,

$$|f(t)| \leq |f(t) - f_N(t)| + |f_N(t)| < 1 + M_N,$$

and so f is bounded on $[0,1]$.

(b) Show that the sequence of functions is *uniformly bounded*. That is, show that there is an M such that

$$|f_n(t)| < M$$

for all $t \in [0,1]$ and all n. **Hint.** Show that for large n, f_n can be bounded essentially by the bound for f.

Solution: Let $M_0 = \sup_{t\in[0,1]} |f(t)|$. By the first part, we know that $M_0 < \infty$. There exists an N such that for all $n \geq N$ and all $t \in [0,1]$,

$$|f_n(t) - f(t)| < 1.$$

BUt then for $n \geq N$ and $t \in [0,1]$,

$$|f_n(t)| \leq |f_n(t) - f(t)| + |f(t)| < 1 + M_0.$$

Now let $M = maxM_0 + 1, M_1., \cdots, M_{N-1}$. Then clearly for any f_n and any $t \in [0,1]$,

$$|f_n(t)| < M.$$

(c) Suppose additionally $f_n \in \mathcal{R}[0,1]$ for all n. Prove or disprove -

$$\lim_{n\to\infty} \int_0^{1-1/n} f_n(t)\, dt = \int_0^1 f(t)\, dt.$$

Hint.

$$\int_0^{1-1/n} f_n(t)\, dt = \int_0^1 f_n(t)\, dt - \int_0^{1/n} f_n(t)\, dt$$

Solution: As in the hint,

$$\int_0^{1-1/n} f_n(t)\, dt = \int_0^1 f_n(t)\, dt - \int_0^{1/n} f_n(t)\, dt.$$

Since $f_n \xrightarrow{u.c} f$,

$$\lim_{n\to\infty} \int_0^{1-1/n} f_n(t)\, dt = \int_0^1 f(t)\, dt - \lim_{n\to\infty} \int_0^{1/n} f_n(t)\, dt.$$

To compute the second term we estimate

$$\Big| \int_0^{1/n} f_n(t)\, dt \Big| \leq \int_0^{1/n} |f_n(t)|\, dt \leq \frac{M}{n} \to 0$$

as $n \to \infty$. This completes the proof.

8. Find the radius and interval of convergence of the following series.

1. $\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n}$

Solution: We can rewrite this as

$$\sum_{n=1}^{\infty} \frac{2^n}{n} \left(x + \frac{1}{2}\right)^n.$$

Written this way, the center is clearly $a = -1/2$. To find the radius of convergence we compute

$$\limsup_{n\to\infty} \left(\frac{2^n}{n}\right)^{1/n} = 2,$$

and so $R = 1/2$. So the series definitely converges on $(-1, 0)$. Next, we check boundary points.

- $x = -1$. The series is then
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$
which converges by the alternating series test.

- $x = 0$. The series is then the harmonic series which diverges.

So $I = [-1, 0)$.

2. $\sum_{n=1}^{\infty} \frac{2^n}{n^2} x^n$.

Solution: Apply root test to conclude that radius of convergence is $1/2$ and interval of convergence is $[-1/2, 1/2]$.

3. $\sum_{n=0}^{\infty} \frac{2^n}{n!} x^n$.

Solution: Apply ratio test to conclude that radius of convergence is infinity and hence $I = \mathbb{R}$.

9. Decide whether each proposition is true or false, providing a complete proof, or a counter example, as appropriate.

(a) If $\sum f_n$ converges uniformly, then f_n converges uniformly to zero.

Solution: True. Apply uniform Cauchy criteria for series with $n = m + 1$.

(b) If $0 \le f_n(x) \le g_n(x)$ and $\sum g_n$ converges uniformly, then $\sum f_n$ also converges uniformly.

Solution: True. Again uniform Cauchy criteria for series.

(c) If $\sum f_n$ converges uniformly on E, then there exists constants M_n such that $|f_n(x)| < M_n$ for all $x \in E$ and $\sum M_n$ converges.

Solution: Not necessarily true. Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by
$$f_n(x) = \begin{cases} \frac{1}{n}, & x \in [n, n+1) \\ 0, & \text{otherwise} \end{cases}$$
Then for any $n > m$ and for any $x \in \mathbb{R}$,
$$\sum_{k=m}^{n} f_k(x) < \frac{1}{m},$$
and so the series satisfies uniform Cauchy criteria, and hence is uniformly convergent. But $\sup_{\mathbb{R}} |f_n| = n^{-1}$ and $\sum n^{-1}$ diverges.

(d) If each f_n is uniformly continuous on E and $f_n \xrightarrow{u.c} f$ on E, then f is also uniformly continuous on E.

Solution: True. Given any $\varepsilon > 0$, let $N \in \mathbb{N}$ such that

$$\sup_E |f_N(x) - f(x)| < \frac{\varepsilon}{3}.$$

Since f_N is uniformly continuous, there exists $\delta > 0$ such that for any $x, y \in E$,

$$|x - y| < \delta \implies |f_N(x) - f_N(y)| < \frac{\varepsilon}{3}.$$

So if $x, y \in E$ with $|x - y| < \delta$, then

$$\begin{aligned} |f(x) - f(y)| &\le |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

So f is also uniformly continuous.

(e) If f_n has a finite number of discontinuities on E and $f_n \xrightarrow{u.c} f$, then f has a finite number of discontinuities on E.

Solution: False. See example in 4(a).

(f) If f_n has at most M number of discontinuities on E (where M is fixed and independent of n) and $f_n \xrightarrow{u.c} f$, then f has at most M number of discontinuities on E.

Solution: This is true. To prove, this suppose f has at least $M+1$ discontinuities $p_1, \cdots, p_{M+1}$. We first have the following observation, which follows from a simple counting argument.

Claim. There exists a $k \in \{1, 2, \cdots, M+1\}$ such that for all $N \in \mathbb{N}$, there exists $n \ge N$ such that f_n is continuous at p_k.

Proof If not, then for all $k \in \{1, 2, \cdots, M+1\}$, there exists an $N_k \in \mathbb{N}$ such that for all $n \ge N_k$, f_n is discontinuous at p_k. Let $N = \max(N_1, \cdots, N_{M+1})$. Then f_N is discontinuous at each $p_1, p_2, \cdots, p_{M+1}$ contradicting the hypothesis that f_N can have at most M discontinuities. □

Without loss of generality, let p_1 be such that for all $N \in \mathbb{N}$, there exists $n \ge N$ such that f_n is continuous at p_1. Then there exists a subsequence $\{f_{n_j}\}$ such that each f_{n_j} is continuous at p_1 To see this, simply apply the claim to pick an $n_1 \ge 1$ such that f_{n_1} is continuous at p_1, and then having chosen $n_1 < n_2 < \cdots < n_{j-1}$, apply claim with $N = n_{j-1}$, to obtain $n_j > n_{j-1}$ such that f_{n_j} is continuous at p_1. But then since $f_n \xrightarrow{u.c} f$, we also must have $f_{n_j} \xrightarrow{u.c} f$. This contradicts the theorem on uniform convergence and continuity, since each f_{n_j} is continuous at p_1 but f is discontinuous at p_1.

10. Define

$$g(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{x^{2n} + 1}.$$

Find the values of x where the series converges, and show that we get a continuous function on this set.

Solution: Let $f_n(x) = \frac{x^{2n}}{1+x^{2n}}$. If $|x| \ge 1$, clearly $\lim_{n\to\infty} f_n(x) \ne 0$. On the other hand, for any $x \in [-r, r]$ with $r < 1$, since $1 + x^{2n} > 1$, we have

$$|f_n(x)| \le r^{2n}.$$

And since $\sum r^{2n}$ converges, by the Weierstrass M-test, the series converges uniformly on any subset $[-r, r] \subset (-1, 1)$. Since each function f_n is continuous on $(-1, 1)$, by the theorem on uniform convergence and continuity, $g(x)$ is defined and continuous on $(-1, 1)$.

11. Let

$$h(x) = \sum_{n=1}^{\infty} \frac{1}{x^2 + n^2}.$$

(a) Show that h is continuous on all of $\mathbb{R}$.

Solution: For any $x \in \mathbb{R}$,

$$\left| \frac{1}{x^2 + n^2} \right| \leq \frac{1}{n^2},$$

and since $\sum n^{-2}$ converges, by the Weierstrass M-test, the given series converges uniformly to $h(x)$ on all of $\mathbb{R}$. Since each term is continuous on $\mathbb{R}$ and the convergence is uniform, the infinite sum $h(x)$ is also continuous on $\mathbb{R}$.

(b) Is h differentiable? If so, is the derivative h' continuous? Give complete proofs.

Solution: Let

$$s_n = \sum_{k=1}^{n} \frac{1}{x^2 + k^2}.$$

We need to verify the two hypothesis in the theorem on uniform convergence and differentiation.

- We have already seen that $s_n(x) \to h(x)$ on all of $\mathbb{R}$.
- We need to check that $s_n'(x)$ converges uniformly on $\mathbb{R}$. We calculate

$$s_n'(x) = -\sum_{k=1}^{n} \frac{2x}{(x^2 + n^2)^2}.$$

For any $x \in \mathbb{R}$, it is easy to see that $2n|x| \leq x^2 + n^2$, and so

$$\left| \frac{2x}{x^2 + n^2} \right| \leq \frac{1}{n(x^2 + n^2)} \leq \frac{1}{n^3}.$$

Since $\sum n^{-3}$ is convergent, again by Weierstrass M-test, $s_n'(x)$ converges uniformly.

Hence by applying the theorem on uniform convergence of series and differentiation (Theorem 6.10 in Week-6 notes), we conclude that h is differentiable on $\mathbb{R}$ and

$$h'(x) = -\sum_{n=1}^{\infty} \frac{2x}{(x^2 + n^2)^2}.$$

Also, we have already seen that the series for $h'(x)$ converges uniformly on $\mathbb{R}$, and so again by the theorem on uniform convergence and continuity, $h'(x)$ is also continuous on $\mathbb{R}$.

12. We saw in class that a function represented by a power series is automatically smooth, that is, it has derivatives of all orders. The aim of this question to show that the converse might not be true. That is, there exist smooth functions that cannot be represented by a power series.

(a) If $P(x)$ is a polynomial, show that

$$\lim_{x\to\infty} P(x)e^{-x^2} = 0.$$

Hint. Use L'Hospital's rule and an induction on the degree of the polynomial.

Solution: It is enough (Why?) to show that for any integer $n \geq 0$,

$$\lim_{t\to\infty} t^n e^{-t} = 0.$$

To prove this we use L'Hospital's rule and induction. The base case $n = 0$ is trivial. Suppose we have prove that the limit is zero for $1, 2, \cdots n-1$. To prove it for n, we notice that $t^n e^{-t} = t^n/e^t$. Since both numerator and denominator converge to infinity, we can use L'Hospital's theorem. So

$$\lim_{t\to\infty} \frac{t^n}{e^t} = \lim_{t\to\infty} \frac{nt^{n-1}}{e^t} = n \lim_{t\to\infty} \frac{t^{n-1}}{e^t}.$$

The final limit is zero because of our inductive hypothesis.

(b) Now define

$$f(t) = \begin{cases} e^{-1/t^2}, \ t \neq 0 \\ 0, \ t = 0. \end{cases}$$

Show that f has derivatives of all orders at $x = 0$, and that $f^{(n)}(0) = 0$ for all $n = 1, 2, \cdots$. Can f be represented by a power series in a neighborhood of $t = 0$?

Solution: Even though we have to find derivatives at $t = 0$, since they are higher order derivatives, we are forced to compute $f^{(n)}(t)$ for $t \neq 0$. We have the following claim.
Claim. For any n, there is a polynomial $P_n(x)$ such that for any $t \neq 0$,

$$f^{(n)}(t) = P_n\left(\frac{1}{t}\right)e^{-1/t^2}.$$

Proof. We proceed by induction. For $n = 0$, simply choose $P_0(x) = 1$. Having constructed, $P_0, P_1, \cdots, P_{n-1}$, we construct P_n. By the inductive hypothesis, for $t \neq 0$,

$$f^{(n-1)}(t) = P_{n-1}\left(\frac{1}{t}\right)e^{-1/t^2}.$$

Differentiating by using chain and product rules,

$$\begin{aligned} f^{(n)}(t) &= P'_{n-1}\left(\frac{1}{t}\right) \cdot \left(\frac{-1}{t^2}\right)e^{-1/t^2} + P_{n-1}\left(\frac{1}{t}\right)\left(\frac{2}{t^3}\right)e^{-1/t^2} \\ &= \left[P_{n-1}\left(\frac{1}{t}\right)\left(\frac{2}{t^3}\right) - P'_{n-1}\left(\frac{1}{t}\right) \cdot \left(\frac{1}{t^2}\right)\right]e^{-1/t^2}. \end{aligned}$$

So if we let

$$P_n(x) = 2P_{n-1}(x)x^3 - P'_{n-1}(x)x^2,$$

then $P_n(x)$ is a polynomial such that for all $t \neq 0$,

$$f^{(n)}(t) = P_n\left(\frac{1}{t}\right)e^{-1/t^2}.$$

Now, coming back to the problem, we compute $f^{(n)}(0)$ by induction. The base case $n = 0$ is trivial since by definition $f(0) = 0$. Suppose now we have shown that $f^{(n-1}(0)$ exists an is zero.

Then,

$$
\begin{aligned}
f^{(n)}(0) &= \lim_{h \to 0} \frac{f^{(n-1)}(h) - f^{((n-1)}(0)}{h} \\
&= \lim_{h \to 0} \frac{1}{h} P_{n-1}\left(\frac{1}{h}\right) e^{-1/h^2} \qquad \text{(by claim)} \\
&= \lim_{k \to \infty} k P_{n-1}(k) e^{-k^2} \\
&= 0
\end{aligned}
$$

where the final equality follows from part(a) and the observation that $kP_{n-1}(k)$ is again a polynomial in k. By definition then the Taylor of f centered at $t = 0$ is trivial, that is

$$T_f(t; 0) = 0$$

for all t. Since $f(t)$ is never zero when $t \neq 0$, it follows that $f(t) \neq T_f(t; 0)$ for all $t \neq 0$, or in other words, even though $f \in C^\infty(\mathbb{R})$, f is not represented by a power series in any neighborhood of $t = 0$.

Periodicity of sine and cosine.

This exercise is unimportant from the point of view of doing well in the course, but highly recommended for those with a wish to explore the non-trivial origins of one of the most trivial of high-school facts.

Recall that $\sin x, \cos x : \mathbb{R} \to \mathbb{R}$ are defined by the power series

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.$$

In class, we saw that as a consequence, we have the following basic properties:

$$\frac{d \sin x}{dx} = \cos x, \quad \frac{d \cos x}{dx} = \sin x$$
$$\sin(-x) = \sin(x), \quad \cos(-x) = \cos(x)$$
$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$
$$\sin^2 x + \cos^2 x = 1.$$

Our aim is to prove the following theorem.

Theorem. There exists a real number $\pi > 0$ such that $\sin(x + 2\pi) = \sin(x)$ for all $x \in \mathbb{R}$. Moreover, if $\beta \in \mathbb{R}$ such that $\sin(x + \beta) = \sin(x)$ for all x, then $\beta = 2n\pi$ for some $n \in \mathbb{Z}$.

Proof.

(i) Show that if $0 \le x \le \sqrt{2}$, then for all $n = 0, 1, 2, \cdots$,

$$\frac{x^{4n}}{(4n)!} - \frac{x^{4n+2}}{(4n+2)!} \ge 0.$$

Hence show that $\cos x > 0$ if $x \in [0, \sqrt{2}]$. In particular, this shows that there is no root of $\cos x$ in $[0, \sqrt{2}]$.

(ii) Show that $\cos 2 < -\frac{1}{3}$, and hence show that there is at least one root of $\cos x$ in $[\sqrt{2}, 2]$.

(iii) Next, show that $\sin x \ge \frac{x}{3}$ when $x \in [0, 2]$. Use this to show that $|\sin x| \ge \frac{|x|}{3}$ for all $x \in [-2, 2]$.

(iv) Use this and the addition formulas to conclude that $\cos x$ has a unique root in $[\sqrt{2}, 2]$. **Hint.** If $\sqrt{2} \le x_1 < x_2 \le 2$ were two roots, then show that $\sin(x_2 - x_1)$ would have to be zero. But this should contradict the above inequality.

(v) Let this unique root be ζ and define $\pi = 2\zeta$. Show that $\cos n\pi = (-1)^n$ for all integers n. In particular, show that $\cos 2n\pi = 1$.

(vi) Hence, show that $\sin(x + 2\pi) = \sin(x)$ and $\cos(x + 2\pi) = \cos(x)$. It now remains to show that any other period has to be an integer multiple of 2π.

(vii) Show that if $\sin(x + \beta) = \sin x$ for all x, then $\sin(\beta/2) = 0$ and hence $\cos \beta = 1$.

(viii) To finish off the proof of the theorem, show that if $\cos \beta = 1$, then $\beta = 2n\pi$ for some integer n. **Hint.** Without loss of generality, let $\beta > 0$. There is a natural number n such that $-\pi \le \beta - 2n\pi < \pi$. Show that $\sin(\beta/2 - n\pi) = 0$ but that this contradicts the inequality in (iii) unless $\beta = 2n\pi$.