Assignment-5

(Due 07/23)

Only submit the questions in red.

1. Suppose f is a bounded real valued function on $[a, b]$ such that $f^2 \in \mathcal{R}[a, b]$. Does it follow that $f \in \mathcal{R}[a, b]$? Does the answer change if we assume $f^3 \in \mathcal{R}[a, b]$? Either give a proof or provide a counter example in each of the two cases.

Solution: No to the first part. Consider

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ -1, & \text{otherwise.} \end{cases}$$

Then $f^2 = 1$ everywhere and so is integrable, but f is discontinuous everywhere and hence is non-integrable. For the second part, the answer is yes. COnsider the continuous function $\varphi(x) = x^{1/3}$. Then if f^3 is integrable, by the theorem on composition, $\varphi \circ f^3 = f$ is also integrable.

Remark. This reasoning does not work for the first part, since if you let $\varphi(x) = x^{1/2}$ (which is continuous), then $\varphi \circ f^2 = |f|$ and not f.

2. Let

$$f(x) = \begin{cases} x^2, & x \in \mathbb{Q} \\ 0, & \text{otherwise.} \end{cases}$$

(a) Calculate the upper and lower integrals $U(f)$ and $L(f)$ for f on $[0, b]$.

Solution: For any partition $\mathcal{P}$ it is clear that $L(\mathcal{P}, f) = 0$ and hence $L(f) = 0$.
Claim. $U(f) = b^2/3$.
Proof. Consider the function, $g(x) = x^2$ for all $x \in [0, b]$. Since rationals are dense, for any partition $\mathcal{P}$, $U(\mathcal{P}, f) = U(\mathcal{P}, g)$. In particular, $U(f) = U(g)$. But g, being continuous, is clearly integrable on $[0, b]$ and so

$$U(g) = \int_0^b x^2 \, dx = \frac{b^3}{3}.$$

Hence $U(f) = b^2/3$. □

(b) Is f integrable on $[0, b]$. Answer the question, solely based on your calculations in part(a), and not by quoting a theorem that we might have learnt in class.

Solution: Since $U(f) \neq L(f)$, f is not integrable.

3. Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(t) = \begin{cases} 2^{-n}, & 2^{-n-1} < t \leq 2^{-n} \\ 0, & t = 0. \end{cases}.$$

Show that $f \in \mathcal{R}[0,1]$ by showing that given any $\varepsilon > 0$, there exists a partition P such that

$$U(P,f) - L(P,f) < \varepsilon,$$

and without appealing to the theorem of Lebesgue.

Solution: Let $\varepsilon > 0$. Consider the interval $[\varepsilon/2, 1]$. Then f has only finitely many discontinuities on this interval, and so by the proof of the Theorem on continuity and integrability discussed in class, there is a partition P' of $[\varepsilon/2, 1]$ such that

$$U(P',f) - L(P',f) < \frac{\varepsilon}{2}.$$

Now let $P = \{0, \varepsilon/2\} \cup P'$. Then P is a partition of the interval $[0,1]$. Let $M = \sup_{t\in[0,\varepsilon/2]} f(t)$ and $m = \inf_{t\in[0,\varepsilon/2]} f(t)$. Clearly, $M < 1$ and $m = 0$. Then

$$U(P,f) - L(P,f) = (M-m)\frac{\varepsilon}{2} + U(P',f) - L(P',f) < \varepsilon.$$

4. (a) Let $f \in \mathcal{R}[a,b]$ and $\{p_1, \cdots p_n\}$ be a finite collection of points in $[a,b]$. Let $g : [a.b] \to \mathbb{R}$ be a bounded function such that

$$f(t) = g(t),$$

for all $t \in [a,b] \setminus \{p_1, \cdots, p_n\}$. Show that $g \in \mathcal{R}[a,b]$ and that

$$\int_a^b g(t)\,dt = \int_a^b f(t)\,dt.$$

Hint. Do it for one point at a time.

Solution: Let us assume that $n = 1$, and denote $p_1 = p$. The general case follows by a repeated use of the argument below. Let us also put $h = f - g$. Since $g = f - h$, it is enough to show that $h \in \mathcal{R}[a,b]$ and that

$$\int_a^b h(t)\,dt = 0.$$

Note that $h(t) = 0$ for all $t \neq p$. If $h(p) = 0$, there is nothing to prove. So suppose $h(p) \neq 0$. For simplicity, let us also assume that $p \in (a,b)$. If p is one of the boundary points, the argument is even easier. Then given $\varepsilon > 0$, let

$$\delta = \min\left(\frac{p-a}{2}, \frac{\varepsilon}{4|h(p)|}, \frac{b-p}{2}\right),$$

and consider the partition,

$$P = \{t_0 = a, t_1 = p - \delta, t_2 = p + \delta, t_3 = b\}.$$

Then

$$U(P,h)2M_2\delta,\ \ L(P,h) = 2m_2\delta.$$

But since $M_2, m_2 \leq |h(p)|$,

$$|U(P,h)|, |L(P,h)| \leq 2|h(p)|\delta < \frac{\varepsilon}{2}.$$

In particular,

$$U(P,h) - L(P,h) < \varepsilon.$$

So for any $\varepsilon > 0$ we have a partition for which the difference in the upper and lower sums is smaller than ε. This shows that h is integrable. Moreover, we know that the integral is sandwiched between the upper and the lower sums and hence it follows from the above estimates that

$$-\varepsilon < \int_a^b h(t)\,dt < \varepsilon.$$

But since this si true for all $\varepsilon > 0$, it forces the integral to be zero.

(b) Is the conclusion true, if we instead have a countable collection of points $\{p_n\}_{n=1}^{\infty}$?

Solution: No. Consider $f(t) = 1$ on $[0,1]$ and

$$g(t) = \begin{cases} 1, & t \in \mathbb{R} \setminus \mathbb{Q} \\ 0, & t \in \mathbb{Q}. \end{cases}$$

5. (a) If $f \in \mathcal{R}[0,1]$, show that

$$\int_0^1 f(t)\,dt = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\Big(\frac{k}{n}\Big).$$

Solution: Let $\varepsilon > 0$. We need to show the following -

Claim. There exists an integer N such that for all $n > N$,

$$\Big| \int_0^1 f(t)\,dt - \frac{1}{n} \sum_{k=1}^{n} f\Big(\frac{k}{n}\Big) \Big| < \varepsilon.$$

Proof. By Theorem 5.1 in the notes, since $f \in \mathcal{R}[0,1]$, there exists a $\delta > 0$ such that for any partition P with $|P| < \delta$, we have that

$$U(P,f) - L(P,f) < \varepsilon.$$

Now consider the partition P_n given by subdividing $[0,1]$ into n subintervals of equal length $1/n$. That is,

$$P_n = \{0, \frac{1}{n}, \frac{2}{n}, \cdots, 1\}.$$

Then cleary if N is an integer such that $N > 1/\delta$, then for any $n > N$ we have that $|P_n| < \delta$. Applying the above consequence of THem 5.1, we see that

$$U(P_n, f) - L(P_n, f) < \varepsilon.$$

But from the definition it follows that

$$L(P_n, f) \le \frac{1}{n} \sum_{k=1}^{n} f\Big(\frac{k}{n}\Big) \le U(P_n, f)$$

$$L(P_n, f) \le \int_a^b f(t)\,dt \le U(P_n, f).$$

The first set of inequalities hold since on any interval $I_k = [\frac{k-1}{n}, \frac{k}{n}]$ if like usual we set $M_k = \sup_{I_k} f(t)$ and $m_k = \inf_{I_k} f(t)$, then

$$m_k \leq f\Big(\frac{k}{n}\Big) \leq M_k.$$

But now since $0 \leq U(P_n, f) - L(P_n, f) < \varepsilon$, the claim is proved from the above two inequalities. Again, it is good to visualize the upper and lower sums as floors of a building, and the integral and the right side approximation sum lie between these two floors.

(b) Give an example of a bounded function $f : [0,1] \to \mathbb{R}$ for which the limit on the right exists, but f is not Riemann integrable.

Solution: Consider

$$f(x) = \begin{cases} 0, & x \in \mathbb{Q} \\ 1, & x \notin \mathbb{Q}. \end{cases}$$

Then the sum on the right is always 0, and hence in particular the limit is also zero, while the function is not Riemann integrable.

(c) Use part(a) to evaluate the limit

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \frac{1}{\sqrt{k}}.$$

Solution: We write

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{n} \frac{1}{\sqrt{k}} = \frac{1}{n} \sum_{k=1}^{n} \frac{\sqrt{n}}{\sqrt{k}} = \frac{1}{n} \sum_{k=1}^{n} f\Big(\frac{k}{n}\Big),$$

where we let $f(x) = \frac{1}{\sqrt{x}}$. Then by part(a),

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \frac{1}{\sqrt{k}} = \int_0^1 \frac{dx}{\sqrt{x}} = 2.$$

6. (a) Let f be a continuous real valued function on $[a, b]$. Show that there exists a $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(t)\, dt.$$

Solution: By the fundamental theorem of calculus

$$F(x) = \int_a^x f(t)\, dt$$

is differentiable on $[a, b]$ and $F'(x) = f(x)$. By the mean value theorem, there exists a $c \in [a, b]$ such that

$$(b-a)F'(c) = F(b) - F(a)$$

whcih is what we want.

(b) More generally, if f is continuous on $[a,b]$, $g \in \mathcal{R}[a,b]$ and g does not change sign (you can assume $g \geq 0$), then prove that exists a $c \in [a,b]$ such that

$$\int_a^b f(t)g(t)\,dt = f(c)\int_a^b g(t)\,dt.$$

Hint. Let $I = \int_a^b g(t)\,dt \neq 0$ and $f[a,b] = [m,M]$. The proof is easy if $I = 0$. If $I \neq 0$, show that

$$m < \frac{1}{I}\int_a^b f(t)g(t)\,dt < M,$$

and use intermediate value theorem. There is also a much neater way to do this using change of variable. Can you figure it out?

Solution: As in the hint, let $I = \int_a^b g(t)\,dt$. Since $m < f(t) < M$, and $g \geq 0$, we have that $mg(t) \leq f(t) \leq Mg(t)$ for all $t \in (a,b)$. Integrating we get

$$mI \leq \int_a^b f(t)g(t)\,dt < MI.$$

If $I = 0$, then it follows that $\int_a^b f(t)g(t)\,dt = 0$, and there is nothing to prove. So suppose $I \neq 0$. Then

$$m \leq \frac{1}{I}\int_a^b \leq M.$$

Then by intermediate value theorem, since f is continuous, it follows that there is some $c \in (a,b)$ such that

$$f(c) = \frac{1}{I}\int_a^b f(t)g(t)\,dt.$$

7. Let $f : [1,\infty) \to (0,\infty)$ be a continuous, decreasing function such that $\lim_{x\to\infty} f(x) = 0$. Denote

$$s_n = \sum_{k=1}^{n} f(k),\ \ I_n = \int_1^n f(t)\,dt,\ \ d_n = s_n - I_n.$$

(a) Show that $f(n) + I_n \leq s_n \leq f(1) + I_n$.

Solution: We can write

$$I_n = \sum_{k=1}^{n-1}\int_k^{k+1} f(x)\,dx.$$

For any $x \in [k,k+1]$,

$$f(k+1) \leq f(x) \leq f(k).$$

Integrating the first inequality from k to $k+1$,

$$f(k+1) \leq \int_k^{k+1} f(x)\,dx,$$

and so

$$\begin{aligned} I_n &\geq \sum_{k=1}^{n-1} f(k+1) \\ &= s_n - f(1), \end{aligned}$$

or equivalently,

$$s_n \leq I_n + f(1).$$

This proves the right side of the inequality. For the left side, we integrate the second inequality above, and obtain that

$$I_n \leq \sum_{k=1}^{n-1} f(k) = s_n - f(n),$$

or equivalently

$$I_n + f(n) \leq s_n.$$

(b) (Integral test for convergence) Hence show that $\sum_{n=1}^{\infty} f(n)$ converges if and only if $\int_1^{\infty} f(x)\,dx$ converges.

Solution: Firstly recall that $\int_1^{\infty} f(x)\,dx$ converges if and only

$$\lim_{R\to\infty} \int_1^R f(x)\,dx$$

exists. Since $f(x) \geq 0$, this limit exists if and only if $\int_1^R f(x)\,dx$ is bounded by a constant independent of R. Or equivalently,

$$\int_1^{\infty} f(x)\,dx \text{ converges} \iff \{I_n\} \text{ is bounded}.$$

Now the conclusion follows from the following chain of equivalences

$$\begin{aligned} \sum f(n) \text{ converges} &\iff \{s_n\} \text{ converges} \\ &\iff \{s_n\} \text{ is bounded} \qquad \text{(since } f(n) \geq 0\text{)} \\ &\iff \{I_n\} \text{ is bounded} \\ &\iff \int_1^{\infty} f(x)\,dx \text{ converges}. \end{aligned}$$

(c) Use the above test, to find all possible values of p and q for which the following series converge.

1. $\sum_{n=2}^{\infty} \frac{1}{n^p (\ln n)^q}$

Solution: There are many cases we need to deal with. Before that we start with the following simple observation, that for any $\alpha > 0$, there exists an $N > 0$ (possibly depending on α) such that

$$\ln(n) < n^{\alpha}$$

for all $n > N$. This follows from the fact that $\lim_{n\to\infty} \ln(n)/n^{\alpha} = 0$, which itself can be proved by an application of L'Hospital. With this out of the way, we have the following three cases.

- $p > 1$. In this case if $q \geq 0$, then since $\ln(n) \geq 1$ for all $n > 3$,

$$\frac{1}{n^p (\ln(n))^q} < \frac{1}{n^p}$$

for $n > 3$. So by comparison theorem, the series converges. If on the other hand, $q < 0$, let $r = -q$. Then $r > 0$, and

$$\frac{1}{n^p(\ln n)^q} = \frac{(\ln n)^r}{n^p}.$$

By the above observation, for any $\alpha > 0$, there exists and N such that for any $n > N$, $(\ln n)^r < n^{\alpha r}$, and so

$$\frac{(\ln n)^r}{n^p} < \frac{1}{n^{p-\alpha r}}.$$

Since $p > 1$, we can choose $\alpha > 0$ small enough so that $p - \alpha r > 1$. For such a choice of α, the series $\sum n^{\alpha r - p}$ converges. But then by the comparison test, the original series converges. To sum up, in this case, the series converges no matter what the value of q is.

- $p = 1$. Here the series reduces to

$$\sum \frac{1}{n(\ln n)^q}.$$

Let $f(x) = 1/x(\ln x)^q$. Then f is a non-negative decreasing function on $[2, \infty)$. By the integral test, the given series converges if and only if $\int_2^\infty f(x)\,dx$ converges. But by change of variables $u = \ln x$, we see that

$$\int_2^R f(x)\,dx = \int_{\ln 2}^{\ln R} \frac{du}{u^q},$$

and so the integral, and hence the series, converges if and only if $q > 1$.

- $p < 1$. In this case if $q \leq 0$, for any $n > 3$,

$$\frac{1}{n^p(\ln n)^q} > \frac{1}{n^p}.$$

Since $\sum n^{-p}$ diverges for $p < 1$, by the comparison test, the given series also diverges. If $q > 0$, then for any $\alpha > 0$, there exists an N such that for any $n > N$, $(\ln n)^q < n^{\alpha q}$, and so

$$\frac{1}{n^p(\ln n)^q} > \frac{1}{n^{p+\alpha q}}.$$

Since $p < 1$, one can choose $\alpha > 0$ small enough so that $p + \alpha q < 1$, and then the series

$$\sum \frac{1}{n^{p+\alpha q}}$$

diverges. Again by comparison test, the original series diverges. So to sum up, if $p < 1$, the series diverges no matter what the value of q.

So finally the series converges if and only if

$$(p, q) \in \{p > 1,\ q \in \mathbb{R}\} \cup \{p = 1, q > 1\}.$$

2. $\sum_{n=3}^{\infty} \frac{1}{n \ln n (\ln \ln n)^p}$.

Solution: Consider the function

$$f(x) = \frac{1}{x \ln x (\ln \ln x)^p}.$$

Clearly it is decreasing and positive on $x \geq 3$. Also, note that if we let $u = \ln \ln x$, then $du = dx/(x \ln x)$, and so by change of variables formula,

$$\int_3^R f(x)\,dx = \int_{\ln \ln 3}^{\ln \ln R} u^{-p}\,du.$$

Clearly the second integral has a limit as $R \to \infty$ if $p > 1$, and so by the integral test, the series converges if and only if $p > 1$.

8. (a) Let $f : [a,b] \to \mathbb{R}$ be continuous function, and let $p, q : [c,d] \to [a,b]$ be continuous functions, differentiable on the interior (c,d). Define

$$F(x) = \int_{p(x)}^{q(x)} f(t)\,dt.$$

Show that F is continuous on $[c,d]$ and differentiable on (c,d), and that

$$F'(x) = f(q(x))q'(x) - f(p(x))p'(x).$$

Hint. Write F as a composition of two functions, one of which can be differentiated using the fundamental theorem of calculus. Then properties of F follow from corresponding properties of compositions.

Solution: Let $h(x)$ be an antiderivative of f. That is, $h'(x) = f(x)$. Then by the second fundamental theorem of calculus,

$$F(x) = h(q(x)) - h(p(x)).$$

By chain rule,

$$\begin{aligned} F'(x) &= h'(q(x))q'(x) - h'(p(x))p'(x) \\ &= f(q(x))q'(x) - f(p(x))p'(x). \end{aligned}$$

(b) Now, let $F : (0, \pi) \to \mathbb{R}$ be defined by

$$F(x) = \int_{\sin x}^{1} \ln t\,dt.$$

Calculate $F'(\pi/4)$ and $F'(\pi/2)$ in two ways. First, by evaluating the integral, and then differentiating. And second, by using part(a) above. Your answers should of course be the same.

Solution:

- **Direct calculation.** By integration by parts, it is easy to see that

$$\int \ln t\,dt = t \ln t - t,$$

and so

$$F(x) = -\sin x \ln(\sin x) + \sin x - 1.$$

Differentiating, we see that

$$F'(x) = -\cos x \ln(\sin x),$$

and so

$$F'\left(\frac{\pi}{4}\right) = \frac{\ln 2}{2\sqrt{2}}, \ F'\left(\frac{\pi}{2}\right) = 0.$$

- **Using part(a).** By the formula in part(a), with $q(x) = 1$ and $p(x) = \sin x$,

$$F'(x) = -\cos x \ln(\sin x),$$

and so we have the same formula as above.

9. (a) Let $f(x) = |x|$, and define $F(x) = \int_{-1}^{x} f(t)\,dt$. Find a piecewise algebraic formula for $F(x)$. Where is F continuous? Where is it differentiable? Where does $F' = f$?

Solution:

- $x < 0$. In that case $|t| = -t$ for all $t \in [-1, x]$ and so

$$F(x) = -\int_{-1}^{x} t\,dt = \frac{1}{2} - \frac{x^2}{2}.$$

- $x \geq 0$. In this case,

$$F(x) = -\int_{-1}^{0} t\,dt + \int_{0}^{x} t\,dt = \frac{1}{2} + \frac{x^2}{2}.$$

So $F(x) = \frac{1+x|x|}{2}$.

(b) Now repeat part(a) with

$$f(x) = \begin{cases} 1, & x < 0 \\ 2, & x \geq 0. \end{cases}$$

Solution:

- $x < 0$. Then

$$F(x) = \int_{-1}^{x} dt = 1 + x.$$

- $x \geq 0$. Then

$$F(x) = \int_{-1}^{0} dt + 2\int_{0}^{x} dt = 1 + 2x.$$

10. Calculate $\lim_{x\to 0} \frac{1}{x}\int_0^x e^{t^2}\,dt$. Give complete justifications, quoting any theorems that might have been used.

Solution: Since e^{t^2} is continuous, the function

$$F(x) = \int_0^x e^{t^2}\, dt$$

is differentiable on $\mathbb{R}$, and moreover $F'(x) = e^{x^2}$. In particular $F'(0) = 1$. On the other hand, by definition,

$$1 = F'(0) = \lim_{x\to 0} \frac{F(x) - F(0)}{x} = \lim_{x\to 0} \frac{1}{x} \int_0^x e^{t^2}\, dt.$$

11. Find the set of all values of p for which the following improper integrals converge.

1. $\int_0^1 \frac{1-\sin x}{x^p}$.

Solution: Since $1 < \pi/2$, clearly, $1-\sin x$ decreases on $[0,1]$ and so $0 < 1-\sin 1 < 1-\sin x \leq 1$ for all $x \in [0,1]$. So

$$\frac{1-\sin 1}{x^p} < \frac{1-\sin x}{x^p} \leq \frac{1}{x^p},$$

and by the comparison test, the given integral converges if and only if $\int_0^1 x^{-p}\, dx$ converges, which means if and only if $p < 1$.

2. $\int_0^\infty \frac{\ln(1+x)}{x^p}$.

Solution: We need to analyze near 0 and infinity separately. That is, we write

$$\int_0^\infty \frac{\ln(1+x)}{x^p} = \int_0^1 \frac{\ln(1+x)}{x^p} + \int_1^\infty \frac{\ln(1+x)}{x^p}.$$

- For the first integral, the Taylor series for

$$\ln(1+x) = x - \frac{x^2}{2} + \cdots,$$

indicates that $\ln(1+x)/x^p$ behaves like $x^{-(p-1)}$ near zero, and so should be integrable near zero, if $p-1 < 1$ or $p < 2$. To check this rigorously, we need the following.

Claim. There exists a $\delta > 0$ such that for all $x \in (0,\delta)$,

$$\frac{x}{2} < \ln(1+x) < \frac{3x}{2}.$$

Proof. By L'Hospital's rule

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1,$$

and so there exists $\delta > 0$ such that for all $x \in (0,\delta)$ such that

$$\left|\frac{\ln(1+x)}{x} - 1\right| < \frac{1}{2},$$

or equivalently,

$$\frac{1}{2} < \frac{\ln(1+x)}{x} < \frac{3}{2},$$

and the claim follows. □

Having proved the claim, it follows by the comparison theorem, that $\int_0^1 \frac{\ln(1+x)}{x^p}$ converges if and only if $\int_0^1 x^{1-p}$ converges, which happens if and only if $p < 2$.

- To analyze the integral at infinity, we again use the basic fact that given any $\alpha > 0$, there exists $M = M(\alpha)$ such that $\ln(1+x) < x^\alpha$ for all $x > M$. If $p > 1$, let $\alpha > 0$ such that $p - \alpha > 1$. Then since for all $x > M(\alpha)$,

$$\frac{\ln(1+x)}{x^p} \leq x^{\alpha - p},$$

and $\int_1^\infty x^{\alpha-p}$ converges, it follows by the comparison principle that $\int_1^\infty \frac{\ln(1+x)}{x^p}$ converges. On the other hand, if $p \leq 1$, then

$$\frac{\ln(1+x)}{x} \geq \frac{1}{x^p}.$$

Since $\int_1^\infty x^{-p}$ diverges, it follows that the given integral also diverges.

Combining the two cases above, we see that the integral is convergent if and only if $1 < p < 2$.

Hint. Taylor's theorem might be useful while analyzing the integrands near $x = 0$.

12. (a) Show that for $n = 1, 2, 3, \cdots$

$$\int_{(n-1)\pi}^{n\pi} \left| \frac{\sin x}{x} \right| dx \geq \frac{2}{n\pi}.$$

Solution: Since $1/x > 1/n\pi$ on the given region, and that $\sin x > 0$ on $[0, \pi]$ and periodic with period 2π,

$$\begin{aligned} \int_{(n-1)\pi}^{n\pi} \left| \frac{\sin x}{x} \right| dx &\geq \frac{1}{n\pi} \int_{(n-1)\pi}^{n\pi} |\sin x| \\ &= \frac{1}{n\pi} \int_0^\pi \sin x \, dx \\ &= \frac{2}{n\pi}. \end{aligned}$$

(b) Hence show that $\int_\pi^\infty \left| \frac{\sin x}{x} \right| dx$ diverges.

Solution: For any $m \in \mathbb{N}$,

$$\int_\pi^{m\pi} \left| \frac{\sin x}{x} \right| dx = \sum_{n=2}^{m} \int_{(n-1)\pi}^{n\pi} \left| \frac{\sin x}{x} \right| dx \geq \frac{2}{\pi} \sum_{n=2}^{m} \frac{1}{n} \xrightarrow{m\to\infty} \infty.$$

(c) For any $R > 0$, show that

$$\int_\pi^R \frac{\sin x}{x} \, dx = -\frac{1}{\pi} - \frac{\cos R}{R} + \int_\pi^R \frac{\cos x}{x^2} \, dx.$$

Solution: Integrating by parts with $u = 1/x$ and $dv = -d\cos x$,

$$\int_\pi^R \frac{\sin x}{x}\,dx = \Big[-\frac{\cos x}{x}\Big]_\pi^R + \int_\pi^R \frac{\cos x}{x^2}\,dx = -\frac{1}{\pi} - \frac{\cos R}{R} + \int_\pi^R \frac{\cos x}{x^2}\,dx.$$

(d) Hence, show that $\int_\pi^\infty \frac{\sin x}{x}\,dx$ is a convergent integral.

Solution: By the above part,

$$\lim_{R\to\infty} \int_\pi^R \frac{\sin x}{x}\,dx = -\frac{1}{\pi} + \lim_{R\to\infty} \int_\pi^R \frac{\cos x}{x^2}\,dx.$$

To show that the integral converges, we need to prove that the limit on the right exists and is finite. Note that

$$\Big|\frac{\cos x}{x^2}\Big| \le \frac{1}{x^2},$$

and so since $\int_\pi^\infty x^{-2}$ converges, by the comparison test, $\int_\pi^\infty |\cos x/x|$, and hence $\int_\pi^\infty \cos x/x$ converges. This shows that the limit on the right also exists and hence the original integral is convergent.

Remark In fact, one can show that

$$\int_0^\infty \frac{\sin x}{x} = \frac{\pi}{2}.$$

This formula is usually one of the high points of a course in complex analysis, and is a consequence of the so-called residue theorem (yes, a real integral evaluated using complex numbers!). But there are are several, real-variable proofs of this, including using doubles integrals or using differentiation under the integral sign (popularized by Feynman, as an alternative to residue calculus).