Assignment-4

(not to be handed in)

1. Show that if f is differentiable at $x = p$, then
$$\lim_{h\to 0} \frac{f(p+h) - f(p-h)}{2h} = f'(p).$$

Solution: Follows from the observation that
$$\frac{f(p+h) - f(p-h)}{2h} = \frac{f(p+h) - f(p)}{2h} + \frac{f(p) - f(p-h)}{2h},$$
and that
$$\lim_{h\to 0} \frac{f(p) - f(p-h)}{h} = \lim_{k\to 0} \frac{f(p+k) - f(p)}{k} = f'(p),$$
which can be seen by setting $k = -h$.

2. Let f and g be differentiable functions on (a, b) and let $p \in (a, b)$. Define
$$h(t) = \begin{cases} f(t), & t \in (a, p) \\ g(t), & t \in [p, b). \end{cases}$$
Show that h is differentiable on (a, b) if and only if $f(p) = g(p)$ and $f'(p) = g'(p)$.

Solution:

- $\Longrightarrow$**.** h is continuous, and so $f(p) = h(p+) = h(p-) = g(p)$. In particular, $h(p) = f(p) = g(p)$. Now, let
$$\varphi(t) = \frac{h(t) - h(p)}{t - p},$$
be the difference quotient of h. Then
$$\varphi(p+) = \lim_{t\to p^+} \frac{h(t) - h(p)}{t - p} = \frac{f(t) - f(p)}{t - p} = f'(p).$$
Similarly, $\varphi(p-) = g'(p)$, and since h is differentiable, $\varphi(p+) = \varphi(p-)$ and so $f'(p) = g'(p)$.
- $\Longleftarrow$**.** Now suppose $f(p) = g(p)$ and $f'(p) = g'(p)$. Then in particular, $h(p) = f(p) = g(p)$. SO if $\varphi(t)$ is the difference quotient of h as above, then again, we can see that $\varphi(p+) = f'(p)$ and $\varphi(p-) = g'(p)$. So by the hypothesis, $\varphi(p+) = \varphi(p-)$, and the $\lim_{t\to p} \varphi(t)$ exists. Hence h is differentiable.

3. (a) Show that $|\sin\theta| \le |\theta|$, for all $\theta \in \mathbb{R}$.

Solution: Special case of part(b) below.

(b) More generally, show that if $g : \mathbb{R} \to \mathbb{R}$ is differentiable such that $|g'(t)| \le M$ and $g(0) = 0$, then

$$|g(t)| \le M|t|,$$

for all $t \in \mathbb{R}$.

Solution: Let $t \in \mathbb{R}$ and $t \neq 0$. Then by the mean value theorem, since $g(0) = 0$, there exists a c between 0 and t such that

$$g(t) = g'(c)t.$$

Taking absolute value,

$$|g(t)| = |g'(c)||t| \le M|t|.$$

4. (a) Show that $\tan x > x$ for all $x \in (0, \pi/2)$.

Solution: Consider the function $f(x) = \tan x - x$. Then

$$f'(x) = \sec^2 x - 1 > 0,$$

if $x \in (0, \pi/2)$. So the function is increasing on the given region. But $f(0) = 0$, and so $f(x) > 0$ on $(0, \pi/2)$.

(b) Show that

$$\frac{2x}{\pi} < \sin x < x$$

for all $x \in [0, \pi/2]$. **Hint.** Consider the function $\sin x/x$. Is it monotonic?

Solution: As in the hint, consider

$$f(x) = \begin{cases} \sin x/x, x \in (0, \pi/2] \\ 1,\ \ x = 0. \end{cases}$$

Clearly f is continuous on $[0, \pi/2]$. For $x \in (0, \pi/2)$,

$$f'(x) = \frac{x\cos x - \sin x}{x^2}.$$

By part(a),

$$\frac{\sin x}{\cos x} > x,$$

and so (since $\cos x > 0$), we see that $f'(x) < 0$ for all $x \in (0, \pi/2)$. So the function is decreasing and

$$f(\pi/2) \le f(x) \le f(0),$$

which gives us the required inequalities.

5. Find the following limits if they exist.

(a) $\lim_{x\to 0} \frac{x - \sin x}{x^3}$

Solution: Applying L'Hospital's rule twice (or actually thrice),

$$\lim_{x\to 0}\frac{x-\sin x}{x^3} = \lim_{x\to 0}\frac{\frac{d(x-\sin x)}{dx}}{\frac{dx^3}{dx}} = \lim_{x\to}\frac{1-\cos x}{3x^2} = \frac{1}{3}\lim_{x\to 0}\frac{\sin x}{2x} = \frac{1}{6}.$$

(b) $\lim_{x\to 0}\frac{1-\cos 2x-2x^2}{x^4}$

Solution: One can again apply L'Hospital's rule two times. Instead, we use Taylor's theorem. Letting, $f(x)=\cos(2x)$, we see that

$$f(0)=1,\ f'(0)=0,\ f''(0)=-4,\ f^{(3)}(0)=0,\ f^{(4)}(0)=16,$$

and so by Taylor's theorem,

$$\cos(2x) = 1-2x^2+\frac{2}{3}x^4-\frac{32\sin(2c)}{5!}x^5,$$

for some c between 0 and x. But since $|\sin\theta|\le 1$, we see that

$$\left|\frac{1-\cos 2x-2x^2}{x^4}+\frac{2}{3}\right| \le \frac{32}{5!}|x|.$$

By squeeze principle, letting $x\to 0$, we see that

$$\lim_{x\to 0}\frac{1-\cos 2x-2x^2}{x^4} = -\frac{2}{3}.$$

(c) $\lim_{x\to\infty}(e^x+x)^{1/x}$

Solution:

- **Method-1.** Let $y=(e^x+x)^{1/x}$. Then
$$\ln y = \frac{\ln(e^x+x)}{x}.$$
By L'Hospital,
$$\lim_{x\to\infty}\frac{\ln(e^x+x)}{x} = \lim_{x\to\infty}\frac{e^x+1}{e^x+x} = 1.$$
So $\ln y \xrightarrow{x\to\infty} 1$. Exponentiating both sides, since e^x is continuous, $y=e^{\ln y}\to e^1$, and so
$$\lim_{x\to\infty}(e^x+x)^{1/x} = e.$$
- **Method-2.** Note that
$$(e^x+x)^{1/x} = e(1+xe^{-x})^{1/x} = e(1+xe^{-x})^{e^{-x}/xe^{-x}} = e\Big[(1+xe^{-x})^{1/xe^{-x}}\Big]^{e^{-x}}.$$
Now let $y=xe^{-x}$. Then $(e^x+x)^{1/x} = e[(1+y)^{1/y}]^{e^{-x}}$ Clearly, $\lim_{x\to\infty} y = 0$. Also,
$$\lim_{y\to 0}(1+y)^{1/y} = e.$$
And so, by the theorem on limits of compositions,
$$\lim_{x\to\infty}(e^x+x)^{1/x} = e[\lim_{y\to 0}(1+y)^{1/y}]^0 = e.$$

(d) $\lim_{x\to 0}(\cos x)^{1/x^2}$.

Solution: Again, let $y = (\cos x)^{1/x^2}$. Then

$$\ln y = \frac{\ln \cos x}{x^2},$$

and so

$$\lim_{x\to 0} \ln y = -\lim_{x\to 0} \frac{\sin x}{2x\cos x} = -\frac{1}{2}\lim_{x\to 0}\frac{\sin x}{x}\cdot \lim_{x\to 0}\frac{1}{\cos x} = -\frac{1}{2},$$

and so $\lim_{x\to 0} y = \frac{1}{\sqrt{e}}$.

(e) $\lim_{x\to 0^+} \frac{1-\cos x}{e^x - 1}$

Solution: By L'Hospital

$$\lim_{x\to 0^+} \frac{1-\cos x}{e^x-1} = \lim_{x\to 0^+}\frac{\sin x}{e^x} = 0.$$

(f) $\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right)$

Solution: Again by L'Hospital's

$$\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x\to 0}\frac{x-\sin x}{x\sin x} = \lim_{x\to 0}\frac{1-\cos x}{\sin x + x\cos x} = \lim_{x\to 0}\frac{\sin x}{2\cos x + x\sin x} = 0.$$

6. Consider the functions

$$f(x) = x + \cos x \sin x \text{ and } g(x) = e^{\sin x}(x + \cos x\sin x).$$

(a) Show that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$.

Solution: Note that

$$x - 1 \le f(x),\ e^{-1}(x-1) \le g(x),$$

for all $x \ge 0$. Then by the squeeze princinple we see that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$..

(b) Show that if $\cos x \neq 0$ and $x > 3$, then

$$\frac{f'(x)}{g'(x)} = \frac{2e^{-\sin x}\cos x}{2\cos x + f(x)}.$$

Solution: Simple computation using chain and product rules.

(c) Show that

$$\lim_{x\to\infty}\frac{2e^{-\sin x}\cos x}{2\cos x + f(x)} = 0,$$

and yet, the limit $\lim_{x\to} \frac{f(x)}{g(x)}$ does not exist.

Solution: Clearly,

$$|2e^{-\sin x}\cos x| \le 2e,$$

for all $x \in \mathbb{R}$. Next,

$$2\cos x + f(x) \ge f(x) - 2 \ge x - 3,$$

for all $x > 3$. And so for $x > 3$,

$$\left|\frac{2e^{-\sin x}\cos x}{2\cos x + f(x)}\right| \le \frac{2e}{x-3} \to 0$$

as $x \to \infty$. This proves that

$$\lim_{x\to\infty} \frac{2e^{-\sin x}\cos x}{2\cos x + f(x)} = 0.$$

On the other hand,

$$\frac{f(x)}{g(x)} = e^{-\sin x}$$

which clearly does not have a limit as $x \to \infty$.

(d) Explain why this does not contradict L'Hospital's rule.

Solution: One of the assumptions when using L'Hospital's rule when computing $\lim_{x\to s} f(x)/g(x)$ is that $f'(x)/g'(x)$ is well defined for all points near s, which means in particular that $g'(x) \neq 0$ for all x close enough to s. But in the example above,

$$g'(x) = e^{\sin x}\cos x[2\cos x + f(x)].$$

Consider the sequence $x_n = n\pi/2$. Then $x_n \xrightarrow{n\to\infty} \infty$ and $g'(x_n) = 0$ for all n, and so L'Hospital's rule cannot be applied.

7. (a) Show that $e^x \ge 1 + x$ for all $x \ge 0$ (In the earlier version this was $x \in \mathbb{R}$, which is clearly incorrect).

Solution: Let $f(x) = e^x - 1 - x$, Then $f'(x) = e^x - 1 \ge 0$ for all $x \in \mathbb{R}$. So f is increasing on $\mathbb{R}$. Since $f(0) = 0$, this shows that $x \ge 0 \implies f(x) \ge 0$.

(b) Show that there exists a constant $M > 0$ such that

$$\left|\frac{e^x - 1 - x}{x^2} - \frac{1}{2}\right| \le M|x|,$$

for all $x \in [-1,1] \setminus \{0\}$. **Hint.** Taylor's thoerem.

Solution: By Taylor's theorem, for any $x \in [-1,1]$ and $x \neq 0$, there exists c between x and 0 such that

$$e^x = 1 + x + \frac{x^2}{2} + \frac{e^c}{3!}x^3,$$

and so

$$\left|\frac{e^x - 1 - x}{x^2} - \frac{1}{2}\right| \le M|x|,$$

where we can take $M = e/6$.

(c) Compute

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}.$$

Solution: By squeeze theorem, letting $x \to 0$ in the above estimate, clearly,

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \frac{1}{2}.$$

8. Show the following *Bernoulli's* inequalities.

(a) If $r \in [0, 1]$ and $x \geq -1$, show that

$$(1 + x)^r \leq 1 + rx.$$

Solution: Consider $f(x) = 1 + rx - (1 + x)^r$. Then

$$f'(x) = r\Big[1 - \frac{1}{(1 + x)^{1-r}}\Big].$$

Note that $1 - r \geq 0$. So on $x \geq 0$, clearly $f'(x) \geq 0$ and the function is increasing. On the other hand when $x \in [-1, 0]$ clearly $f'(x) \leq 0$. This shows that the function decreases on $[-1, 0]$ and increases on $[0, \infty)$, and so the 0 is a minima. Since $f(0) = 0$, this shows that for all $x \in [-1, \infty)$, $f(x) \geq 0$.

(b) If $r \in (-\infty, 0) \cup (1, \infty)$, and $x \geq -1$, show that

$$(1 + x)^r \geq 1 + rx.$$

Solution: This time consider the function $f(x) = (1 + x)^r - rx - 1$. Then

$$f'(x) = r\Big[(1 + x)^{r-1} - 1\Big].$$

Now there are two cases.

- $r \in (-\infty, 0)$. In this case if $x \in [-1, 0]$, $(1+x)^{r-1} - 1 \geq 0$ and if $x > 0$, $(1+x)^{r-1} - 1 \leq 0$. But since $r < 0$ this implies that $f'(x) \leq 0$ if $x \in [-1, 0]$ and $f'(x) \geq 0$ if $x > 0$. So 0 is clearly the minimum point, and since $f(0) = 0$, we have that $f(x) \geq 0$.
- $r \in [1, \infty)$. Here when $x \in [-1, 0]$ we see that $(1 + x)^{r-1} - 1 \leq 0$ and if $x > 0$, $(1 + x)^{r-1} - 1 \geq 0$. But now since $r > 0$, we again have that $f'(x) \leq 0$ if $x \in [-1, 0]$ and $f'(x) \geq 0$ if $x > 0$. And so once again 0 is clearly the minimum point, and since $f(0) = 0$, we have that $f(x) \geq 0$.

Hint. You can either use the try to find the local max or min, or simply use the fact that if $f' \geq 0$, then f is increasing.

9. Suppose $f \in C^5[-1, 1]$, such that $f(0) = 1$, and $f'(0) = \cdots = f^4(0) = 0$. If $f^5(0) < 0$, show that there exists a $\delta > 0$ such that

$$f(x) < 1,$$

for all $x \in (0, \delta)$.

Solution: Since $f^{(5)}(x)$ is continuous, and since $f^{(5)}(0) < 0$, there is a $\delta > 0$ such that $f^{(5)}(x) < 0$ for all $x \in (0, \delta)$. Now by Taylor's theorem, for any $x \in (0, \delta)$, there exists a $c_x \in (0, x)$ such that

$$\begin{aligned} f(x) &= f(0) + f'(0)x + \frac{f^{(2)}(0)}{2}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(2)}(c_x)}{5!}x^5 \\ &= 1 + \frac{f^{(2)}(c_x)}{5!}x^5, \end{aligned}$$

since$f(0) = 1$ and $f'(0) = \cdots = f^4(0) = 0$. Now since $c_x \in (0, \delta)$, $f^{(5)}(c_x) < 0$ and $x^5 > 0$ for $x \in (, \delta)$ we have that

$$f(x) < 1$$

for all $x \in (0, \delta)$.

10. A function $f : E \to \mathbb{R}$ is called *Lipschitz* (or more precisely M-Lipschitz) if there exists an $M > 0$ such that for all $x, y \in E$,

$$|f(x) - f(y)| \le M|x - y|.$$

(a) Show that any Lipschitz function is uniformly continuous.

Solution: Given $\varepsilon > 0$, simply let $\delta = \varepsilon/M$ in the definition of uniform continuity,

(b) Show that if $f : (a, b) \to \mathbb{R}$ is a differentiable function such that $|f'(t)| \le M$ for all $t \in (a, b)$, then f is M-Lipschitz.

Solution: Follows from the mean value theorem.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be a *contraction*, that is an α-Lipschitz function, for some $\alpha < 1$. Show that there exists a *fixed point* p, that is, a $p \in \mathbb{R}$ such that $f(x) = x$.

Solution: Let $x_0 \in \mathbb{R}$ be any real number. Having chosen $x_0, x_1, \cdots, x_n$, let $x_{n+1} = f(x_n)$.
Claim-1. $\{x_n\}$ is a Cauchy sequence.
Proof. Without loss of generality, we can assume that $x_1 = f(x_0) \neq x_0$, or else x_0 would be a fixed point, and we are already done. Since f is a contraction,

$$|x_{n+1} - x_n| = |f(x_n) - f(x_{n-1})| \le \alpha|x_n - x_{n-1}|.$$

Applying this inductively, we see that

$$|x_{n+1} - x_n| \le \alpha^n|x_1 - x_0|.$$

So for any $m > n$, by repeated use of triangle inequality,

$$\begin{aligned} |x_m - x_n| &\le |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n| \\ &\le (\alpha^{m-1} + \alpha^{m-2} + \cdots + \alpha^n)|x_1 - x_0| \\ &\le (\alpha^n + \alpha^{n+1} + \cdots)|x_1 - x_0| \\ &= \frac{\alpha^n}{1 - \alpha}|x_1 - x_0|, \end{aligned}$$

where we used the fact that since $\alpha < 1$, the corresponding geometric series is convergent and has sum $1/(1-\alpha)$. Now, given any $\varepsilon > 0$, let N be such that

$$\alpha^N < \frac{\varepsilon(1-\alpha)}{|x_1 - x_0|}.$$

This can be done since $\lim_{N\to\infty} \alpha^N = 0$. Then for $m > n > N$, by the above estimate,

$$|x_m - x_n| \leq \frac{\alpha^n}{1-\alpha}|x_1 - x_0| < \frac{\alpha^N}{1-\alpha}|x_1 - x_0| < \varepsilon.$$

This proves that the sequence is Cauchy. □

Since $\{x_n\}$ is Cauchy, it is also convergent, and we denote $\lim_{n\to\infty} x_n = p$.

Claim-2. $f(p) = p$.

Proof. Consider the equation $x_{n+1} = f(x_n)$. Since f is Lipshitz, it is in particular, continuous. And so taking limits on both sides,

$$p = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = f(p).$$

(d) Show that the fixed point so obtained will be unique.

Solution: If there are two fixed points p and q, such that $p \neq q$, then

$$|p - q| = |f(p) - f(q)| \leq \alpha|p - q|,$$

which is a contradiction since $\alpha < 1$.

11. A function $f : E \to \mathbb{R}$ is said to be *α-Hölder* for $\alpha > 0$, if

$$|f(x) - f(y)| \leq M|x - y|^\alpha,$$

for all $x, y \in E$ and some $M > 0$.

(a) Show that any α-Hölder function is uniformly continuous.

Solution: Given $\varepsilon > 0$, simply pick $\delta = (\varepsilon/M)^{1/\alpha}$ in the definition of uniform continuity.

(b) Show that if $f : (a, b) \to \mathbb{R}$ is α-Hölder for some $\alpha > 1$, then f is differentiable, and is in fact a constant function.

Solution: Let $x \in (a, b)$ and $\varphi(t)$ be the difference quotient at x. Then

$$|\varphi(t)| = \left|\frac{f(t) - f(x)}{t - x}\right| \leq M|t - x|^{\alpha-1} \xrightarrow{t\to x} 0.$$

since $\alpha - 1 > 0$. Hence, not only is f differentiable on (a, b), but in fact $f'(x) = 0$ for all x. Hence f must be a constant.

12. Assume that f has a finite derivative on (a, ∞).

(a) If $f(x) \to 1$ and $f'(x) \to c$ as $x \to \infty$, prove that $c = 0$. **Hint.** Show, using the mean value theorem, that there is a sequence $x_n \in (n, n+1)$ such that $f'(x_n) \to 0$.

Solution: By the mean value theorem, for each n, there exists a $x_n \in [n, n+1]$ such that

$$f'(x_n) = f(n+1) - f(n).$$

Since $\lim_{x\to\infty} f'(x) = c$, it follows that and $\lim_{n\to\infty} f'(x_n) = c$. Also

$$\lim_{n\to\infty} f(n) = \lim_{n\to\infty} f(n+1) = 1.$$

So taking limit as $n \to \infty$ we see that $c = 0$.

(b) If $f'(x) \to 1$ as $x \to \infty$, prove that

$$\lim_{x\to\infty} \frac{f(x)}{x} = 1.$$

Solution: We use L'Hospital's rule. To do that, we need to show that the numerator $f(x) \to \infty$ as $x \to \infty$.

Since $f'(x) \to 1$ as $x \to \infty$, there exists a K such that for all $x > K$, $f'(x) > 1/2$. We will choose $N > K$. By the mean value theorem, there exists a $c \in [K, x]$ (depending possibly on x) such that

$$f(x) = f(K) + (x - K)f'(c) > f(K) + \frac{x-K}{2}.$$

Now letting $x \to \infty$, we see that $f(x) \to \infty$. Now applying L'Hospital's rule,

$$\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} f'(x) = 1.$$