Solutions to Assignment-3

1. (a) Let $f : (a, b) \to \mathbb{R}$ be continuous such that for some $p \in (a, b)$, $f(p) > 0$. Show that there exists a $\delta > 0$ such that $f(x) > 0$ for all $x \in (p - \delta, p + \delta)$.

> **Solution:** Let $\varepsilon > 0$ such that $f(p) - \varepsilon > 0$ (for instance one can take $\varepsilon = f(p)/2$). Since f is continuous, there exists $\delta > 0$ such that
>
> $$|x - p| < \delta \implies |f(x) - f(p)| < \varepsilon.$$
>
> In particular, for all $x \in (p - \delta, p + \delta)$, $f(x) > f(p) - \varepsilon > 0$.

(b) Let $E \subset \mathbb{R}$ be a subset such that there exists a sequence $\{x_n\}$ in E with the property that $x_n \to x_0 \notin E$. Show that there is an unbounded continuous function $f : E \to \mathbb{R}$.

> **Solution:** Consider the function
>
> $$f(x) = \frac{1}{x - x_0}.$$
>
> Since $x_0 \notin E$, this function is continuous on E. On the other hand, by the hypothesis, $\lim_{n\to\infty} |f(x_n)| = \infty$, and so the function is unbounded on E.

2. (a) If $a, b \in \mathbb{R}$, show that

$$\max\{a, b\} = \frac{(a + b) + |a - b|}{2}.$$

> **Solution:** If $a \leq b$, then $\max\{a, b\} = b$.

(b) Show that if $f_1, f_2, \cdots, f_n$ are continuous functions on a domain $E \subset \mathbb{R}$, then

$$g(x) = \max\{f_1(x), \cdots, f_n(x)\}$$

is again a continuous function on E.

> **Solution:** For $n = 2$, use part(a) to write
>
> $$g(x) = \frac{(f_1(x) + f_2(x)) + |f_1(x) - f_2(x)|}{2}.$$
>
> Since $f_1 + f_2$ and $|f_1 - f_2|$ are continuous, it follows that g is also continuous. For $n > 2$ and $k = 2, 3, \cdots, n$, let
>
> $$g_k(x) = \max(f_1(x) \cdots, f_k(x)).$$
>
> In particular $g_n = g$. We use induction to show that $g_k(x)$ is continuous for all $k = 2, \cdots, n$. The base case $k = 2$ is verified, since we have already shown that $g_2(x)$ is continuous. For the inductive step, suppose $g_{k-1}(x)$ is continuous. We note that
>
> $$g_k(x) = \max(g_{k-1}(x), f_k(x)),$$

and again by the above argument for max of two continuous functions, we see that $g_k(x)$ is also continuous. By induction $g_n(x) = g(x)$ is also continuous.

(c) Let's explore if the infinite version of this true or not. For each $n \in \mathbb{N}$, define

$$f_n(x) = \begin{cases} 1, & |x| \geq 1/n \\ n|x|, & |x| < 1/n. \end{cases}$$

Explicitly compute $h(x) = \sup\{f_1(x), f_2(x), \cdots, f_n(x), \cdots\}$. Is it continuous?

Solution: For any $x \neq 0$, there exists an N such that $|x| > 1/n$ for all $n > N$ and $|x| \leq 1/n$ for $n \leq N$, and so $f_n(x) = 1$ for all $n > N$ and for $n \leq N$, $f_n(x) = n|x| \leq 1$. On the other hand, $f_n(0) = 0$ for all n, and hence

$$h(x) = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0, \end{cases}$$

and is discontinuous.

3. For each of the following, decide if the function is uniformly continuous or not. In either case, give a proof using just the definition in terms of ε and δ.

(a) $f(x) = \sqrt{x^2 + 1}$ on $(0, 1)$.

Solution: Note that

$$\begin{aligned} |f(x) - f(y)| &= |\sqrt{x^2+1} - \sqrt{y^2+1}| \\ &= \frac{|x^2 - y^2|}{\sqrt{x^2+1} + \sqrt{y^2+1}} \\ &= \frac{|x-y||x+y|}{\sqrt{x^2+1} + \sqrt{y^2+1}}. \end{aligned}$$

Now if $x, y \in (0, 1)$, then $|x + y| < 2$, and moreover $x^2 + 1, y^2 + 1 \geq 1$, and so

$$|f(x) - f(y)| < 4|x - y|.$$

Given $\varepsilon > 0$, let $\delta = \varepsilon/4$. Then

$$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

(b) $g(x) = x \sin(1/x)$ on $(0, 1)$.

Solution: **Note.** As was mentioned by some students in class, this problem does not seem to have a solution without an appeal to the mean value theorem (MVT), which we of course did not cover last week. Below is the most canonical attempt towards a solution, and you will see the point at which I dont think one can proceed without MVT. For an independent proof of uniform continuity, without actually using showing the dependence of δ on ε, simply consider the function $G : [0, 1] \to \mathbb{R}$,

$$G(x) = \begin{cases} x \sin(1/x), & x \in (0, 1] \\ 0, & x = 0. \end{cases}$$

We have shown in class that this function is continuous on $[0,1]$. Since $[0,1]$ is closed and bounded, $G(x)$ is uniformly continuous. But then $G(x) = g(x)$ on $(0,1)$, and so $g(x)$ is also uniformly continuous.

Failed attempt at a solution.

$$\begin{aligned}\left|(x+h)\sin\left(\frac{1}{x+h}\right) - x\sin\left(\frac{1}{x}\right)\right| &\le |x|\left|\sin\left(\frac{1}{x+h}\right) - \sin\left(\frac{1}{x}\right)\right| + |h|\left|\sin\left(\frac{1}{x+h}\right)\right| \\ &\le |x|\left|\sin\left(\frac{1}{x+h}\right) - \sin\left(\frac{1}{x}\right)\right| + |h|.\end{aligned}$$

For the first term, we use the fact that

$$\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right),$$

and so

$$\begin{aligned}|x|\left|\sin\left(\frac{1}{x+h}\right) - \sin\left(\frac{1}{x}\right)\right| &= 2|x|\left|\sin\left(\frac{h}{2x(x+h)}\right)\cos\left(\frac{2x+h}{2x(x+h)}\right)\right| \\ &\le 2|x|\left|\sin\left(\frac{h}{2x(x+h)}\right)\right|.\end{aligned}$$

At this point, we really need the fact that $|\sin\theta| \le |\theta|$ for all θ, and I don't know any proof of this without using the mean value theorem. This inequality also follows from the fact that differentiable functions with non-negative derivatives are increasing, but this latter fact itself is a consequence of the mean value theorem!

(c) $g(x) = \frac{1}{x^2}$ on $[1,\infty)$.

Solution: If $x, y \ge 1$, then

$$|g(x) - g(y)| = \frac{|x-y|(x+y)}{x^2y^2} = \left(\frac{1}{xy^2} + \frac{1}{yx^2}\right)|x-y| < 2|x-y|.$$

So given $\varepsilon > 0$, let $\delta = \varepsilon/2$ in the definition of uniform continuity.

(d) $g(x) = \frac{1}{x^2}$ on $(0,1]$

Solution: The function is not uniformly continuous. Consider the sequences

$$x_n = \frac{1}{n},\ y_n = \frac{1}{2n}.$$

Then $|x_n - y_n| = 1/2n < 1/n$. On the other hand,

$$|g(x_n) - g(y_n)| = \frac{1}{y_n^2} - \frac{1}{x_n^2} = 3n^2 > 3,$$

if $n > 1$. This contradicts the definition of uniform continuity for $\varepsilon = 3$.

4. (a) Let $f : E \to \mathbb{R}$ be uniformly continuous. If $\{x_n\}$ is a Cauchy sequence in E, show that $\{f(x_n)\}$ is also a Cauchy sequence.

Solution: Let $\varepsilon > 0$. Since f is uniformly continuous, there exists $\delta > 0$ such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

Since $\{x_n\}$ is Cauchy, there exists N such that for all $m, n > N$,

$$|x_n - x_m| < \delta.$$

Combining the two, if $n, m > N$, then

$$|f(x_n) - f(x_m)| < \varepsilon.$$

Since this works for all $\varepsilon > 0$, $\{f(x_n)\}$ is Cauchy.

(b) Show, by exhibiting an example, that the above statement is not true if f is merely assumed to be continuous.

Solution: Let $f(x) = \sin(1/x)$. Clearly $f(x)$ is continuous on $(0, 1)$. But consider the sequence

$$x_n = \frac{2}{n\pi}.$$

Since $x_n \to 0$, it is clearly Cauchy. But

$$f(x_n) = \begin{cases} 0, & n \text{ is even} \\ (-1)^{\frac{n-1}{2}}, & n \text{ is odd,} \end{cases}$$

and hence the sequence $\{f(x_n)\}$ is not Cauchy.

(c) Let $f : (a, b) \to \mathbb{R}$ be continuous. Show that there exists a continuous function $F : [a, b] \to \mathbb{R}$ such that $F(x) = f(x)$ for all $x \in (a, b)$ if and only if f is uniformly continuous. **Hint.** Given f, how should you define $F(a)$ and $F(b)$?

Solution: Consider the sequence $x_n = a + 1/n$. For large enough n, $a_n \in (a, b)$. Since $\{a_n\}$ is Cauchy, and since f is uniformly continuous, by part(a), $\{f(a_n)\}$ is Cauchy, and hence converges. Let

$$A = \lim_{n\to\infty} f(a_n).$$

Similarly, consider $b_n = b - 1/n$ and define

$$B = \lim_{n\to\infty} f(b_n),$$

and define

$$F(x) = \begin{cases} A, & x = a \\ f(x), & x \in (a, b) \\ B, & x = b. \end{cases}$$

Clearly F is an extension of f.
Claim. F is continuous on $[a, b]$.
Proof. Clearly F is continuous on (a, b). To prove continuity at a, let $\{x_n\}$ be a sequence in (a, b) converging to a. We need to show that $F(x_n) = f(x_n) \to F(a) = A$. Let $\varepsilon > 0$. There exists N_1 such that for all $n > N_1$,

$$|A - f(a_n)| < \frac{\varepsilon}{2}.$$

The proof will be complete if we can show that for n large enough $|f(x_n)-f(a_n)|$ can be made smaller than $\varepsilon/2$. This is where we use uniform continuity. By uniform continuity of f in (a,b), there exists a $\delta > 0$ such that

$$|x-y| < \delta \implies |f(x)-f(y)| < \frac{\varepsilon}{2}.$$

Now, since $x_n \to a$ and $a_n = a + 1/n$, there exists N_2 such that for all $n > N_2$,

$$|x_n - a_n| < \delta,$$

and hence for all $n > N_2$,

$$|f(x_n)-f(a_n)| < \frac{\varepsilon}{2}.$$

Letting $N = \max(N_1, N_2)$, using triangle inequality, we see that if $n > N$, then

$$|f(x_n)-A| \leq |f(x_n)-f(a_n)| + |f(a_n)-A| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

5. (a) Show directly from the definition of uniform continuity, that any uniformly continuous function $f : (a,b) \to \mathbb{R}$ is bounded.

Solution: There exists $\delta > 0$ such that for any $x, y \in (a,b)$

$$|x-y| < 2\delta \implies |f(x)-f(y)| < 1.$$

Let $p = (b+1)/2$, that is p is the midpoint of (a,b). The argument actually works for any fixed point in the interval (a,b). Let m be the first natural number such that $p + m\delta \geq b$, and consider the intervals,

$$(a, p-(m-1)\delta],\ [p-(m-1)\delta, p-(m-2)\delta], \cdots,\ [p-\delta, p], [p, p+\delta], \cdots, [p+(m-1)\delta, b).$$

Then any x belongs to at least one of the intervals. Moreover, for any x, y in the *same* interval, $|x-y| < 2\delta$. By triangle inequality, if $x > p$ and $x \in [p+(j-1)\delta, j\delta]$, then

$$\begin{aligned} |f(x)-f(p)| &\leq |f(x)-f(p+(j-1)\delta)| + |f(p+(j-1)\delta) - f(p+(j-2)\delta)| + \cdots + |f(p+\delta)-f(p)| \\ &\leq 1 + \cdots + 1 = j \\ &\leq m \end{aligned}$$

We can use a similar argument for $x < p$. Then by triangle inequality,

$$|f(x)| \leq |f(p)| + m,$$

for all $x \in (a,b)$, and hence the function is bounded.

Note. This also follows directly from $4(c)$ above. Since f is uniformly continuous, there is a continuous extension $F : [a,b] \to \mathbb{R}$. Since $[a,b]$ is closed and bounded, and F is continuous, by extremum value theorem, F is bounded on $[a,b]$. But since $F(x) = f(x)$ for all $x \in (a,b)$ this shows that f is bounded on (a,b).

(b) If $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous, show that there exist $A, B \in \mathbb{R}$ such that $|f(x)| \leq A|x| + B$ for all $x \in \mathbb{R}$. **Hint.** Again apply the definition of uniform continuity with $\varepsilon = 1$. For the corresponding $\delta > 0$, note that any $x \in \mathbb{R}$ can be reached from 0 be a sequence of roughly $|x|/\delta$ steps. Now apply the triangle inequality repeatedly to compare $|f(x)|$ with $|f(0)|$.

Solution: The solution is similar to the one above. By uniform continuity, there exists $\delta > 0$ such that

$$|y - x| < 2\delta \implies |g(y) - g(x)| < 1.$$

Claim. For any real numbers a and b and non negative n such that $|b - a| = n\delta$, we have

$$|f(b) - f(a)| \leq n.$$

Proof. Without loss of generality, we can assume $a < b$ and so $b = a + n\delta$. for some positive integer n. If $n = 0$, there is nothing to prove, so we can assume $n > 0$. Then

$$\begin{aligned} |f(b) - f(a)| &\leq |f(b) - f(b-\delta)| + |f(b-\delta) - f(b-2\delta)| + \cdots + |f(b-(n-1)\delta) - f(a)| \\ &= \sum_{k=0}^{n-1} |f(b-k\delta) - f(b-(k-1)\delta)| \\ &\leq n. \end{aligned}$$

To see the inequality in the third line, apply the above consequence of uniform continuity to $x = b - k\delta$, $y = b - (k-1)\delta$ (so that $|x - y| = \delta < 2\delta$). □

Continuing with the problem, let x be an real number. Then there is an integer m (positive or negative) such that $m\delta \leq x < (m+1)\delta$. In particular, since $|x - m\delta| =< \delta$,

$$|f(x) - f(m\delta)| < 1.$$

On the other hand applying the claim to $a = 0$, $b = m\delta$ and $n = |m|$.

$$|f(m\delta) - f(0)| < |m|.$$

So by triangle inequality, we obtain

$$|f(x) - f(0)| < 1 + |m|.$$

On the other hand, since $m\delta \leq x < (m+1)\delta$, it is easy to see that $|m| < \delta^{-1}|x| + 1$. Using this and triangle inequality, we see that

$$\begin{aligned} |f(x)| &\leq |f(x) - f(0)| + |f(0)| \\ &\leq 1 + |f(0)| + |m| \\ &\leq 2 + |f(0)| + \frac{|x|}{\delta} \\ &\leq A|f(x)| + B, \end{aligned}$$

with $B = 2 + |f(0)|$ and $A = \delta^{-1}$.

6. Let $f : [0,1] \to \mathbb{R}$ be continuous with $f(0) = f(1)$.
   (a) Show that there must exist $x, y \in [0,1]$ satisfying $|x - y| = 1/2$ such that $f(x) = f(y)$.

Solution: As in the hint, consider the function $g(x) = f(x + 1/2) - f(x)$ on $[0, 1/2]$. Then

$$\begin{aligned} g(0) &= f\left(\frac{1}{2}\right) - f(0) \\ g\left(\frac{1}{2}\right) &= f(1) - f\left(\frac{1}{2}\right) = -g(0). \end{aligned}$$

since $f(0) = f(1)$. Either $g(0) = 0$ (and we take $x_0 = 0$), or g changes sign between 0 and 1/2. In the latter case, by intermediate value theorem, there is an $x_0 \in (0, 1/2)$ such that $g(x) = 0$. In either case, if $y = x_0 + 1/2$, then $f(x) = f(y)$.

(b) Show that for each $n \in \mathbb{N}$, there exist $x_n, y_n \in [0, 1]$ such that $|x_n - y_n| = 1/n$ and $f(x_n) = f(y_n)$.

Solution: Now consider

$$g(x) = f(x + 1/n) - f(x)$$

on $[0, \frac{n-1}{n}]$. Since $f(0) = f(1)$, it is easy to see that

$$g(0) + g\Big(\frac{1}{n}\Big) + g\Big(\frac{2}{n}\Big) + \cdots + g\Big(\frac{n-1}{n}\Big) = 0,$$

and so all the terms cannot be of the same sign. That is, either one of $g(k/n) = 0$ (in which case we let $x_0 = k/n$)) or there exists $j < k$ such that $g(j/n)$ and $g(k/n)$ are of opposite signs. Then the intermediate value theorem implies that there is an $x_0 \in (j/n, k/n)$ such that $g(x_0) = 0$. In either case, if $y = x_0 + 1/n$, then $f(x) = f(y)$.

(c) On the other hand, if $h \in [0, 1/2]$ is not of the form $1/n$, show that there does not necessarily exist x, y such that $|x - y| = h$ with $f(x) = f(y)$. Give an example with $h = 2/5$.

Solution: (Due to Rahul) Consider the function

$$f(x) = \cos(5\pi x) + 2x.$$

Clearly $f(0) = f(1) = 1$. One can check easily that

$$f\Big(x + \frac{2}{5}\Big) - f(x) = \frac{4}{5},$$

and hence there is no x such that $f(x + 2/5) = f(x)$.

7. For each stated limit, and ε, find the largest possible δ-neighborhood that makes the definition of limits work.

(a) $\lim_{x \to 4} \sqrt{x} = 2, \ \varepsilon = 1$.

Solution: We need to find the set of all x, such that $|\sqrt{x} - 2| < 1$, or equivalently,

$$-1 < \sqrt{x} - 2 < 1,$$

or $x \in (1, 9)$. Taking $\delta = \min(|4 - 1|, |9 - 14) = 3$, we see that

$$|x - 4| < 3 \implies |\sqrt{x} - 2| < 1,$$

and moreover, this is the largest possible δ.

(b) $\lim_{x \to \pi} \lfloor x \rfloor = 3, \ \varepsilon = 0.01$.

Solution: For any x, either $\lfloor x \rfloor$, which would happen if and only $x \in [3, 4)$, or $|\lfloor x \rfloor - 3| \geq 1$. Since we need $|\lfloor x \rfloor - 3| < 0.01$, this is only possible if $x \in [3, 4)$. But we also want $|x - \pi| < \delta$.

The largest possible δ such that $x \in [3, 4)$ for all x such that $|x - \pi| < \delta$ is given by $\delta = \min(\pi - 3, 4 - \pi) = \pi - 3$.

8. Compute each limit or state that it does not exist. Use any of the tools to justify your answer.

   (a) $\lim_{x \to 2} \frac{|x-2|}{x-2}$.

   Solution: Since
   $$\frac{|x-2|}{x-2} = \begin{cases} 1, & x > 2 \\ -1, & x < 2, \end{cases}$$
   we see that
   $$\lim_{x \to 2^=} \frac{|x-2|}{x-2} \neq \lim_{x \to 2^-} \frac{|x-2|}{x-2} = -1$$

   (b) $\lim_{x \to 0} \sqrt[3]{x}(-1)^{\lfloor 1/x \rfloor}$

   Solution: For any x,
   $$0 \leq |\sqrt[3]{x}(-1)^{\lfloor 1/x \rfloor}| \leq \sqrt[3]{x} \xrightarrow{x \to 0} 0.$$
   By squeeze theorem, $\lim_{x \to 0} \sqrt[3]{x}(-1)^{\lfloor 1/x \rfloor} = 0$.

9. Recall that every rational number x can be written as m/n, where $n > 0$ and $gcd(m, n) = 1$. When $x = 0$, we take $m = 0$ and $n = 1$. Consider $f : \mathbb{R} \to \mathbb{R}$ defined by
   $$f(x) = \begin{cases} 0, & x \text{ is irrational} \\ \frac{1}{n}, & x = \frac{m}{n}. \end{cases}$$

   (a) Show that for any real number α and and integer N, there exists a $\delta > 0$ such that every rational number in the interval $(\alpha - \delta, \alpha + \delta)$, not equal to α, has denominator greater than N. **Hint.** First show that the number of rational numbers in $(\alpha - 1, \alpha + 1)$ with denominator smaller than N is finite. Then choose $\delta < 1$ small enough to exclude all these rationals.

   Solution: We'll pick $\delta < 1$. Consider the rational numbers with denominator smaller than N, that is the denominator (which by our convention is positive) is an integer from the set $\{1, 2, \cdots, N\}$. Any such rational number has a bound
   $$\frac{|m|}{N} < \left|\frac{m}{n}\right| < |m|.$$
   But the rational numbers have to be in the interval $(\alpha - \delta, \alpha + \delta)$ and so in particular in the interval $(\alpha - 1, \alpha + 1)$. That is,
   $$-|\alpha| - 1 < \left|\frac{m}{n}\right| < |\alpha| + 1.$$
   Combining with the above inequalities, we see that m has to satisfy,
   $$-|\alpha| - 1 < |m| < N(|\alpha| + 1).$$
   Since m is an integer, this only leaves a finitely many choices, say $m_1, m_2, \cdots m_K$. So in all we only have finitely many rational numbers $r_1, \cdots r_L$ such that

1. If α is rational, then $r_k \neq \alpha$ for all k.

2. denominator of r_k is smaller than N.

3. $r_k \in (\alpha - 1, \alpha + 1)$.

Let

$$\delta_0 = \frac{1}{2} \cdot \min(|r_1 - \alpha|, \cdots, |r_L - \alpha|),$$

and let $\delta = \min(\delta_0, 1)$ (so that $\delta < 1$ as promised earlier; this was needed in the argument). The clearly $\delta > 0$. Also, if r is a rational in $(\alpha - \delta, \alpha + \delta)$, then the denominator of r has to be bigger than N, and this completes the proof.

(b) For any real number α, show that $\lim_{t \to \alpha} f(t) = 0$.

Solution: Let $\epsilon > 0$, and N be the integer such that $N > 1/\epsilon$. By the Lemma, corresponding to this N, there exists a δ such that for any rational number in $t = m/n$ such that $0 < |\alpha - t| < \delta$ satisfies $n > N$. But then $0 < f(t) = 1/n < 1/N < \epsilon$. On the other hand, for any irrational number, t, $f(t) = 0$ and so for any real number $t \neq \alpha$ such that $|t - \alpha| < \delta$, we have that $|f(t)| < \epsilon$, completing the proof

(c) Prove that f is continuous at every irrational number, and has a removable discontinuity at every rational number.

Solution: If α is an irrational number, then continuity follows from part(b) and the fact that $f(\alpha) = 0$. If α is rational, then part(b) implies that $f(\alpha+)$ and $f(\alpha-)$ exist and are zero, but $f(\alpha) \neq 0$. So f has a removable discontinuity at α.

10. Suppose a and c are real numbers, $c > 0$, and $f : [-1, 1] \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} x^a \sin(|x|^{-c}) \ (x \neq 0), \\ 0 \ (x = 0). \end{cases}$$

Prove the following statements.

(a) f is continuous if and only if $a > 0$.

Solution: Clearly $\lim_{x \to 0} f(x)$ exists if and only $a > 0$, and in that case the limit is in fact 0 and so the function is continuous.

(b) $f'(0)$ exists if and only if $a > 1$.

Solution: Let $\varphi_0(x)$ be the difference quotient at 0. Then

$$\varphi_0(x) = \frac{f(x) - f(0)}{x} = x^{a-1} \sin |x|^{-c}.$$

Now $f'(0)$ exists if and only if $\lim_{x \to 0} \varphi_0(x)$ exists, which by the first part happens if and only if $a - 1 > 0$. Note also that in this case (that is, when $a > 1$), it follows that $f'(0) = 0$.

(c) $f'(x)$ is bounded if and only if $a \geq 1 + c$.

Solution: When $x \neq 0$, we compute $f'(x)$. By chain and product rules

$$f'(x) = ax^{a-1}\sin\left(|x|^{-c}\right) + x^a\cos(|x|^{-c})(-c)|x|^{-c-1}\frac{d|x|}{dx}.$$

Now when $x < 0$, $d|x|/dx = -1$ and when $x > 0$, $d|x|/dx = 1$. So we have that

$$f'(x) = \begin{cases} ax^{a-1}\sin\left(|x|^{-c}\right) - cx^a\cos(|x|^{-c})|x|^{-c-1}, & x > 0 \\ 0, & x = 0 \\ ax^{a-1}\sin\left(|x|^{-c}\right) + cx^a\cos(|x|^{-c})|x|^{-c-1}, & x < 0. \end{cases}$$

Clearly the first terms above are bounded if and only if $a \geq 1$, while the second terms are bounded if and only if $a - c - 1 \geq 0$ or $a \geq c + 1$. Since $c > 0$, $a \geq c + 1$ automatically implies that $a \geq 1$, and so $f'(x)$ is bounded if and only if $a \geq c + 1$.

(d) $f'(x)$ is continuous if and only if $a > 1 + c$.

Solution: Again by the same reasoning as the first part, $\lim_{x\to 0} f'(x)$ exists (and then will equal 0 necessarily) if and only if $a > 1$ and $a > c + 1$. Again, since $c > 0$, this is equivalent to the single inequality $a > c + 1$.