Solutions to Assignment-3

1. (a) Let $f : (a, b) \to \mathbb{R}$ be continuous such that for some $p \in (a, b)$, $f(p) > 0$. Show that there exists a $\delta > 0$ such that $f(x) > 0$ for all $x \in (p - \delta, p + \delta)$.

**Solution:** Let $\varepsilon > 0$ such that $f(p) - \varepsilon > 0$ (for instance one can take $\varepsilon = f(p)/2$). Since $f$ is continuous, there exists $\delta > 0$ such that $|x - p| < \delta \implies |f(x) - f(p)| < \varepsilon$.

In particular, for all $x \in (p - \delta, p + \delta)$, $f(x) > f(p) - \varepsilon > 0$.

(b) Let $E \subset \mathbb{R}$ be a subset such that there exists a sequence $\{x_n\}$ in $E$ with the property that $x_n \to x_0 \notin E$. Show that there is an unbounded continuous function $f : E \to \mathbb{R}$.

**Solution:** Consider the function $f(x) = \frac{1}{x - x_0}$.

Since $x_0 \notin E$, this function is continuous on $E$. On the other hand, by the hypothesis, $\lim_{n \to \infty} |f(x_n)| = \infty$, and so the function is unbounded on $E$.

2. (a) If $a, b \in \mathbb{R}$, show that

$$\max\{a, b\} = \frac{(a + b) + |a - b|}{2}.$$

**Solution:** If $a \leq b$, then $\max\{a, b\} = b$.

(b) Show that if $f_1, f_2, \cdots, f_n$ are continuous functions on a domain $E \subset \mathbb{R}$, then

$$g(x) = \max\{f_1(x), \cdots, f_n(x)\}$$

is again a continuous function on $E$.

**Solution:** For $n = 2$, use part(a) to write

$$g(x) = \frac{(f_1(x) + f_2(x)) + |f_1(x) - f_2(x)|}{2}.$$

Since $f_1 + f_2$ and $|f_1 - f_2|$ are continuous, it follows that $g$ is also continuous. For $n > 2$ and $k = 2, 3, \cdots, n$, let

$$g_k(x) = \max(f_1(x), \cdots, f_k(x)).$$

In particular $g_n = g$. We use induction to show that $g_k(x)$ is continuous for all $k = 2, \cdots, n$. The base case $k = 2$ is verified, since we have already shown that $g_2(x)$ is continuous. For the inductive step, suppose $g_{k-1}(x)$ is continuous. We note that

$$g_k(x) = \max(g_{k-1}(x), f_k(x)),$$
and again by the above argument for max of two continuous functions, we see that \( g_k(x) \) is also continuous. By induction \( g_n(x) = g(x) \) is also continuous.

(c) Let’s explore if the infinite version of this true or not. For each \( n \in \mathbb{N} \), define

\[
f_n(x) = \begin{cases} 1, & |x| \geq 1/n \\ n|x|, & |x| < 1/n. \end{cases}
\]

Explicitly compute \( h(x) = \sup \{f_1(x), f_2(x), \ldots, f_n(x), \ldots \} \). Is it continuous?

**Solution:** For any \( x \neq 0 \), there exists an \( N \) such that \( |x| > 1/n \) for all \( n > N \) and \( |x| \leq 1/n \) for \( n \leq N \), and so \( f_n(x) = 1 \) for all \( n > N \) and for \( n \leq N \), \( f_n(x) = n|x| \leq 1 \). On the other hand, \( f_n(0) = 0 \) for all \( n \), and hence

\[
h(x) = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0, \end{cases}
\]

and is discontinuous.

3. For each of the following, decide if the function is uniformly continuous or not. In either case, give a proof using just the definition in terms of \( \varepsilon \) and \( \delta \).

(a) \( f(x) = \sqrt{x^2 + 1} \) on \((0, 1)\).

**Solution:** Note that

\[
|f(x) - f(y)| = |\sqrt{x^2 + 1} - \sqrt{y^2 + 1}|
= \frac{|x^2 - y^2|}{\sqrt{x^2 + 1} + \sqrt{y^2 + 1}}
= \frac{|x - y||x + y|}{\sqrt{x^2 + 1} + \sqrt{y^2 + 1}}.
\]

Now if \( x, y \in (0, 1) \), then \(|x + y| < 2\), and moreover \( x^2 + 1, y^2 + 1 \geq 1 \), and so

\[|f(x) - f(y)| < 4|x - y|\]

Given \( \varepsilon > 0 \), let \( \delta = \varepsilon/4 \). Then

\[|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon.\]

(b) \( g(x) = x \sin(1/x) \) on \((0, 1)\).

**Solution:** Note. As was mentioned by some students in class, this problem does not seem to have a solution without an appeal to the mean value theorem (MVT), which we of course did not cover last week. Below is the most canonical attempt towards a solution, and you will see the point at which I don’t think one can proceed without MVT. For an independent proof of uniform continuity, without actually using showing the dependence of \( \delta \) on \( \varepsilon \), simply consider the function \( G : [0, 1] \to \mathbb{R}, \)

\[
G(x) = \begin{cases} x \sin(1/x), & x \in (0, 1] \\ 0, & x = 0. \end{cases}
\]
We have shown in class that this function is continuous on $[0, 1]$. Since $[0, 1]$ is closed and bounded, $G(x)$ is uniformly continuous. But then $G(x) = g(x)$ on $(0, 1)$, and so $g(x)$ is also uniformly continuous.

Failed attempt at a solution.

$$
\left| (x+h) \sin \left( \frac{1}{x+h} \right) - x \sin \left( \frac{1}{x} \right) \right| \leq |x| \left| \sin \left( \frac{1}{x+h} \right) - \sin \left( \frac{1}{x} \right) \right| + |h| \left| \sin \left( \frac{1}{x+h} \right) \right|
$$

$$
\leq |x| \left| \sin \left( \frac{1}{x+h} \right) - \sin \left( \frac{1}{x} \right) \right| + |h|.
$$

For the first term, we use the fact that

$$
\sin A - \sin B = 2 \sin \left( \frac{A - B}{2} \right) \cos \left( \frac{A + B}{2} \right),
$$

and so

$$
|x| \left| \sin \left( \frac{1}{x+h} \right) - \sin \left( \frac{1}{x} \right) \right| = 2|x| \left| \sin \left( \frac{h}{2x(x+h)} \right) \cos \left( \frac{2x+h}{2x(x+h)} \right) \right|
$$

$$
\leq 2|x| \left| \sin \left( \frac{h}{2x(x+h)} \right) \right|.
$$

At this point, we really need the fact that $|\sin \theta| \leq |\theta|$ for all $\theta$, and I don’t know any proof of this without using the mean value theorem. This inequality also follows from the fact that differentiable functions with non-negative derivatives are increasing, but this latter fact itself is a consequence of the mean value theorem!

(c) $g(x) = \frac{1}{x^2}$ on $[1, \infty)$.

**Solution:** If $x, y \geq 1$, then

$$
|g(x) - g(y)| = \frac{|x-y|(x+y)}{x^2y^2} = \left( \frac{1}{xy^2} + \frac{1}{y^2x^2} \right) |x-y| < 2|x-y|.
$$

So given $\varepsilon > 0$, let $\delta = \varepsilon/2$ in the definition of uniform continuity.

(d) $g(x) = \frac{1}{x^2}$ on $(0, 1]$

**Solution:** The function is not uniformly continuous. Consider the sequences

$$
x_n = \frac{1}{n}, \ y_n = \frac{1}{2n}.
$$

Then $|x_n - y_n| = 1/2n < 1/n$. On the other hand,

$$
|g(x_n) - g(y_n)| = \frac{1}{y_n^2} - \frac{1}{x_n^2} = 3n^2 > 3,
$$

if $n > 1$. This contradicts the definition of uniform continuity for $\varepsilon = 3$.

4. (a) Let $f : E \to \mathbb{R}$ be uniformly continuous. If $\{x_n\}$ is a Cauchy sequence in $E$, show that $\{f(x_n)\}$ is also a Cauchy sequence.
Solution: Let $\varepsilon > 0$. Since $f$ is uniformly continuous, there exists $\delta > 0$ such that
\[ |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon. \]
Since $\{x_n\}$ is Cauchy, there exists $N$ such that for all $m, n > N$,
\[ |x_n - x_m| < \delta. \]
Combining the two, if $n, m > N$, then
\[ |f(x_n) - f(x_m)| < \varepsilon. \]
Since this works for all $\varepsilon > 0$, $\{f(x_n)\}$ is Cauchy.

(b) Show, by exhibiting an example, that the above statement is not true if $f$ is merely assumed to be continuous.

Solution: Let $f(x) = \sin(1/x)$. Clearly $f(x)$ is continuous on $(0, 1)$. But consider the sequence
\[ x_n = \frac{2}{n\pi}. \]
Since $x_n \to 0$, it is clearly Cauchy. But
\[ f(x_n) = \begin{cases} 0, & n \text{ is even} \\ (-1)^{\frac{n-1}{2}}, & n \text{ is odd}, \end{cases} \]
and hence the sequence $\{f(x_n)\}$ is not Cauchy.

(c) Let $f : (a, b) \to \mathbb{R}$ be continuous. Show that there exists a continuous function $F : [a, b] \to \mathbb{R}$ such that $F(x) = f(x)$ for all $x \in (a, b)$ if and only if $f$ is uniformly continuous. **Hint.** Given $f$, how should you define $F(a)$ and $F(b)$?

Solution: Consider the sequence $x_n = a + 1/n$. For large enough $n$, $a_n \in (a, b)$. Since $\{a_n\}$ is Cauchy, and since $f$ is uniformly continuous, by part(a), $\{f(a_n)\}$ is Cauchy, and hence converges. Let
\[ A = \lim_{n \to \infty} f(a_n). \]
Similarly, consider $b_n = b - 1/n$ and define
\[ B = \lim_{n \to \infty} f(b_n), \]
and define
\[ F(x) = \begin{cases} A, & x = a \\ f(x), & x \in (a, b) \\ B, & x = b. \end{cases} \]
Clearly $F$ is an extension of $f$.

**Claim.** $F$ is continuous on $[a, b]$.

**Proof.** Clearly $F$ is continuous on $(a, b)$. To prove continuity at $a$, let $\{x_n\}$ be a sequence in $(a, b)$ converging to $a$. We need to show that $F(x_n) = f(x_n) \to F(a) = A$. Let $\varepsilon > 0$. There exists $N_1$ such that for all $n > N_1$,
\[ |A - f(a_n)| < \frac{\varepsilon}{2}. \]
5. (a) Show directly from the definition of uniform continuity, that any uniformly continuous function $f : (a, b) \to \mathbb{R}$ is bounded.

**Solution:** There exists $\delta > 0$ such that for any $x, y \in (a, b)$

$$|x - y| < 2\delta \implies |f(x) - f(y)| < 1.$$ 

Let $p = (b + 1)/2$, that is $p$ is the midpoint of $(a, b)$. The argument actually works for any fixed point in the interval $(a, b)$. Let $m$ be the first natural number such that $p + m\delta \geq b$, and consider the intervals

$$(a, p - (m - 1)\delta], [p - (m - 1)\delta, p - (m - 2)\delta], \ldots, [p - \delta, p], [p, p + \delta], \ldots, [p + (m - 1)\delta, b).$$

Then any $x$ belongs to at least one of the intervals. Moreover, for any $x, y$ in the *same* interval, $|x - y| < 2\delta$. By triangle inequality, if $x > p$ and $x \in [p + (j - 1)\delta, j\delta]$, then

$$|f(x) - f(p)| \leq |f(x) - f(p + (j - 1)\delta)| + |f(p + (j - 1)\delta) - f(p + (j - 2)\delta)| + \cdots + |f(p + \delta) - f(p)| \leq 1 + \cdots + 1 = j \leq m$$

We can use a similar argument for $x < p$. Then by triangle inequality,

$$|f(x)| \leq |f(p)| + m,$$

for all $x \in (a, b)$, and hence the function is bounded.

**Note.** This also follows directly from 4(c) above. Since $f$ is uniformly continuous, there is a continuous extension $F : [a, b] \to \mathbb{R}$. Since $[a, b]$ is closed and bounded, and $F$ is continuous, by extremum value theorem, $F$ is bounded on $[a, b]$. But since $F(x) = f(x)$ for all $x \in (a, b)$ this shows that $f$ is bounded on $(a, b)$.

(b) If $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous, show that there exist $A, B \in \mathbb{R}$ such that $|f(x)| \leq A|x| + B$ for all $x \in \mathbb{R}$. **Hint.** Again apply the definition of uniform continuity with $\varepsilon = 1$. For the corresponding $\delta > 0$, note that any $x \in \mathbb{R}$ can be reached from 0 be a sequence of roughly $|x|/\delta$ steps. Now apply the triangle inequality repeatedly to compare $|f(x)|$ with $|f(0)|$.
Solution: The solution is similar to the one above. By uniform continuity, there exists \( \delta > 0 \) such that

\[ |y - x| < 2\delta \implies |g(y) - g(x)| < 1. \]

Claim. For any real numbers \( a \) and \( b \) and non negative \( n \) such that \( |b - a| = n\delta \), we have

\[ |f(b) - f(a)| \leq n. \]

Proof. Without loss of generality, we can assume \( a < b \) and so \( b = a + n\delta \) for some positive integer \( n \). If \( n = 0 \), there is nothing to prove, so we can assume \( n > 0 \). Then

\[
|f(b) - f(a)| \leq |f(b) - f(b - \delta)| + |f(b - \delta) - f(b - 2\delta)| + \cdots + |f(b - (n - 1)\delta) - f(a)| \\
= \sum_{k=0}^{n-1} |f(b - k\delta) - f(b - (k-1)\delta)| \\
\leq n.
\]

To see the inequality in the third line, apply the above consequence of uniform continuity to \( x = b - k\delta \), \( y = b - (k-1)\delta \) (so that \( |x - y| = \delta < 2\delta \)).

Continuing with the problem, let \( x \) be an real number. Then there is an integer \( m \) (positive or negative) such that \( m\delta \leq x < (m+1)\delta \). In particular, since \( |x - m\delta| = \delta < \delta \),

\[ |f(x) - f(m\delta)| < 1. \]

On the other hand applying the claim to \( a = 0 \), \( b = m\delta \) and \( n = |m| \),

\[ |f(m\delta) - f(0)| < |m|. \]

So by triangle inequality, we obtain

\[ |f(x) - f(0)| < 1 + |m|. \]

On the other hand, since \( m\delta \leq x < (m+1)\delta \), it is easy to see that \( |m| < \delta^{-1}|x| + 1 \). Using this and triangle inequality, we see that

\[
|f(x)| \leq |f(x) - f(0)| + |f(0)| \\
\leq 1 + |f(0)| + |m| \\
\leq 2 + |f(0)| + \frac{|x|}{\delta} \\
\leq A|f(x)| + B,
\]

with \( B = 2 + |f(0)| \) and \( A = \delta^{-1} \).

6. Let \( f : [0, 1] \to \mathbb{R} \) be continuous with \( f(0) = f(1) \).

(a) Show that there must exist \( x, y \in [0, 1] \) satisfying \( |x - y| = 1/2 \) such that \( f(x) = f(y) \).

Solution: As in the hint, consider the function \( g(x) = f(x + 1/2) - f(x) \) on \([0, 1/2] \). Then

\[
g(0) = f\left(\frac{1}{2}\right) - f(0) \\
g\left(\frac{1}{2}\right) = f(1) - f\left(\frac{1}{2}\right) = -g(0).
\]
since \( f(0) = f(1) \). Either \( g(0) = 0 \) (and we take \( x_0 = 0 \)), or \( g \) changes sign between \( 0 \) and \( 1/2 \).

In the latter case, by intermediate value theorem, there is an \( x_0 \in (0, 1/2) \) such that \( g(x) = 0 \). In either case, if \( y = x_0 + 1/2 \), then \( f(x) = f(y) \).

(b) Show that for each \( n \in \mathbb{N} \), there exist \( x_n, y_n \in [0, 1] \) such that \( |x_n - y_n| = 1/n \) and \( f(x_n) = f(y_n) \).

**Solution:** Now consider
\[
g(x) = f((x + 1)/n) - f(x)
\]
on \([0, \frac{n-1}{n}]\). Since \( f(0) = f(1) \), it is easy to see that
\[
g(0) + g\left(\frac{1}{n}\right) + g\left(\frac{2}{n}\right) + \cdots + g\left(\frac{n-1}{n}\right) = 0,
\]
and so all the terms cannot be of the same sign. That is, either one of \( g(k/n) = 0 \) (in which case we let \( x_0 = k/n \)) or there exists \( j < k \) such that \( g(j/n) \) and \( g(k/n) \) are of opposite signs. Then the intermediate value theorem implies that there is an \( x_0 \in (j/n, k/n) \) such that \( g(x_0) = 0 \). In either case, if \( y = x_0 + 1/n \), then \( f(x) = f(y) \).

(c) On the other hand, if \( h \in [0, 1/2] \) is not of the form \( 1/n \), show that there does not necessarily exist \( x, y \) such that \( |x - y| = h \) with \( f(x) = f(y) \). Give an example with \( h = 2/5 \).

**Solution:** (Due to Rahul) Consider the function
\[
f(x) = \cos(5\pi x) + 2x.
\]
Clearly \( f(0) = f(1) = 1 \). One can check easily that
\[
f\left(\frac{x + 2}{5}\right) - f(x) = \frac{4}{5},
\]
and hence there is no \( x \) such that \( f(x + 2/5) = f(x) \).

7. For each stated limit, and \( \varepsilon \), find the largest possible \( \delta \)-neighborhood that makes the definition of limits work.

(a) \( \lim_{x \to 4} \sqrt{x} = 2, \ \varepsilon = 1 \).

**Solution:** We need to find the set of all \( x \), such that \( |\sqrt{x} - 2| < 1 \), or equivalently,
\[
-1 < \sqrt{x} - 2 < 1,
\]
or \( x \in (1, 9) \). Taking \( \delta = \min(|4 - 1|, |9 - 14|) = 3 \), we see that
\[
|x - 4| < 3 \implies |\sqrt{x} - 2| < 1,
\]
and moreover, this is the largest possible \( \delta \).

(b) \( \lim_{x \to \pi} [x] = 3, \ \varepsilon = 0.01 \).

**Solution:** For any \( x \), either \([x]\), which would happen if and only \( x \in [3, 4) \), or \([x] - 3 \geq 1 \). Since we need \([x] - 3 < 0.01 \), this is only possible if \( x \in [3, 4) \). But we also want \(|x - \pi| < \delta\).
The largest possible $\delta$ such that $x \in [3, 4)$ for all $x$ such that $|x - \pi| < \delta$ is given by $\delta = \min(\pi - 3, 4 - \pi) = \pi - 3$.

8. Compute each limit or state that it does not exist. Use any of the tools to justify your answer.

(a) $\lim_{x \to 2} \frac{|x - 2|}{x - 2}$.

**Solution:** Since

$$\frac{|x - 2|}{x - 2} = \begin{cases} 1, & x > 2 \\ -1, & x < 2, \end{cases}$$

we see that $\lim_{x \to 2^+} \frac{|x - 2|}{x - 2} \neq \lim_{x \to 2^-} \frac{|x - 2|}{x - 2} = -1$.

(b) $\lim_{x \to 0} \sqrt[3]{x}(-1)^{[1/x]}$

**Solution:** For any $x$,

$$0 \leq |\sqrt[3]{x}(-1)^{[1/x]}| \leq \sqrt[3]{x} \xrightarrow{x \to 0} 0.$$  

By squeeze theorem, $\lim_{x \to 0} \sqrt[3]{x}(-1)^{[1/x]} = 0$.

9. Recall that every rational number $x$ can be written as $m/n$, where $n > 0$ and $gcd(m, n) = 1$. When $x = 0$, we take $m = 0$ and $n = 1$. Consider $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0, & x \text{ is irrational} \\ \frac{1}{n}, & x = \frac{m}{n}. \end{cases}$$

(a) Show that for any real number $\alpha$ and an integer $N$, there exists a $\delta > 0$ such that every rational number in the interval $(\alpha - \delta, \alpha + \delta)$, not equal to $\alpha$, has denominator greater than $N$. **Hint.** First show that the number of rational numbers in $(\alpha - 1, \alpha + 1)$ with denominator smaller than $N$ is finite. Then choose $\delta < 1$ small enough to exclude all these rationals.

**Solution:** We’ll pick $\delta < 1$. Consider the rational numbers with denominator smaller than $N$, that is the denominator (which by our convention is positive) is an integer from the set $\{1, 2, \ldots, N\}$. Any such rational number has a bound

$$\frac{|m|}{N} < \frac{|m|}{n} < |m|.$$  

But the rational numbers have to be in the interval $(\alpha - \delta, \alpha + \delta)$ and so in particular in the interval $(\alpha - 1, \alpha + 1)$. That is,

$$-|\alpha| - 1 < \frac{m}{n} < |\alpha| + 1.$$  

Combining with the above inequalities, we see that $m$ has to satisfy,

$$-|\alpha| - 1 < |m| < N(|\alpha| + 1).$$  

Since $m$ is an integer, this only leaves a finitely many choices, say $m_1, m_2, \ldots m_K$. So in all we only have finitely many rational numbers $r_1, \ldots r_L$ such that
1. If $\alpha$ is rational, then $r_k \neq \alpha$ for all $k$.
2. denominator of $r_k$ is smaller than $N$.
3. $r_k \in (\alpha - 1, \alpha + 1)$.

Let $\delta_0 = \frac{1}{2} \cdot \min(|r_1 - \alpha|, \ldots, |r_L - \alpha|)$,
and let $\delta = \min(\delta_0, 1)$ (so that $\delta < 1$ as promised earlier; this was needed in the argument).

The clearly $\delta > 0$. Also, if $r$ is a rational in $(\alpha - \delta, \alpha + \delta)$, then the denominator of $r$ has to be bigger than $N$, and this completes the proof.

(b) For any real number $\alpha$, show that $\lim_{t \to \alpha} f(t) = 0$.

**Solution:** Let $\epsilon > 0$, and $N$ be the integer such that $N > 1/\epsilon$. By the Lemma, corresponding to this $N$, there exists a $\delta$ such that for any rational number in $t = m/n$ such that $0 < |\alpha - t| < \delta$ satisfies $n > N$. But then $0 < f(t) = 1/n < 1/N < \epsilon$. On the other hand, for any irrational number, $t$, $f(t) = 0$ and so for any real number $t \neq \alpha$ such that $|t - \alpha| < \delta$, we have that $|f(t)| < \epsilon$, completing the proof.

(c) Prove that $f$ is continuous at every irrational number, and has a removable discontinuity at every rational number.

**Solution:** If $\alpha$ is an irrational number, then continuity follows from part(b) and the fact that $f(\alpha) = 0$. If $\alpha$ is rational, then part(b) implies that $f(\alpha^+)$ and $f(\alpha^-)$ exist and are zero, but $f(\alpha) \neq 0$. So $f$ has a removable discontinuity at $\alpha$.

10. Suppose $a$ and $c$ are real numbers, $c > 0$, and $f : [-1, 1] \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} x^a \sin(|x|^{-c}) & (x \neq 0), \\ 0 & (x = 0). \end{cases}$$

Prove the following statements.

(a) $f$ is continuous if and only if $a > 0$.

**Solution:** Clearly $\lim_{x \to 0} f(x)$ exists if and only $a > 0$, and in that case the limit is in fact 0 and so the function is continuous.

(b) $f'(0)$ exists if and only if $a > 1$.

**Solution:** Let $\varphi_0(x)$ be the difference quotient at 0. Then

$$\varphi_0(x) = \frac{f(x) - f(0)}{x} = x^{a-1} \sin |x|^{-c}.$$ 

Now $f'(0)$ exists if and only if $\lim_{x \to 0} \varphi_0(x)$ exists, which by the first part happens if and only if $a - 1 > 0$. Note also that in this case (that is, when $a > 1$), it follows that $f'(0) = 0$.

(c) $f'(x)$ is bounded if and only if $a \geq 1 + c$. 

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Solution: When \( x \neq 0 \), we compute \( f'(x) \). By chain and product rules

\[
f'(x) = ax^{a-1} \sin (|x|^{-c}) + x^a \cos(|x|^{-c})(-c)|x|^{-c-1}\frac{d|x|}{dx}.
\]

Now when \( x < 0 \), \( d|x|/dx = -1 \) and when \( x > 0 \), \( d|x|/dx = 1 \). So we have that

\[
f'(x) =
\begin{cases} 
ax^{a-1} \sin (|x|^{-c}) - cx^a \cos(|x|^{-c})|x|^{-c-1}, & x > 0 \\
0, & x = 0 \\
ax^{a-1} \sin (|x|^{-c}) + cx^a \cos(|x|^{-c})|x|^{-c-1}, & x < 0.
\end{cases}
\]

Clearly the first terms above are bounded if and only if \( a \geq 1 \), while the second terms are bounded if and only if \( a - c - 1 \geq 0 \) or \( a \geq c + 1 \). Since \( c > 0 \), \( a \geq c + 1 \) automatically implies that \( a \geq 1 \), and so \( f'(x) \) is bounded if and only if \( a \geq c + 1 \).

(d) \( f'(x) \) is continuous if and only if \( a > 1 + c \).

Solution: Again by the same reasoning as the first part, \( \lim_{x \to 0} f'(x) \) exists (and then will equal 0 necessarily) if and only if \( a > 1 \) and \( a > c + 1 \). Again, since \( c > 0 \), this is equivalent to the single inequality \( a > c + 1 \).