Solutions to Assignment-2

Only submit the questions in red.

1. (a) For any two sequences \( \{a_n\} \) and \( \{b_n\} \) show that

\[
\limsup_{n \to \infty} (a_n + b_n) \leq \limsup_{n \to \infty} a_n + \limsup_{n \to \infty} b_n,
\]

unless the right hand side is of the form \( \infty - \infty \).

**Solution:** Assume both the limsups are finite (the other cases are also similar). Let \( A = \limsup_{n \to \infty} a_n \), \( B = \limsup_{n \to \infty} b_n \) and \( L = \limsup_{n \to \infty} (a_n + b_n) \). Suppose \( L > A + B \).

Choose an \( \varepsilon > 0 \) such that \( L - \varepsilon > A + B + \varepsilon \). For any \( N > 0 \) there exists an \( n > N \) such that

\[
a_n + b_n > L - \varepsilon. \tag{0.1}
\]

On the other hand, there exists \( N_1 \) such that for all \( n > N_1 \),

\[
a_n < A + \frac{\varepsilon}{2},
\]

and there exists \( N_2 \) such that for all \( n > N_2 \),

\[
b_n < B + \frac{\varepsilon}{2}.
\]

But then if \( N = \max(N_1, N_2) \), then for any \( n > N \) we have

\[
a_n + b_n < A + B + \varepsilon < L - \varepsilon,
\]

contradicting (0.1).

(b) Find sequences \( \{a_n\} \) and \( \{b_n\} \) with strict inequality above.

**Solution:** Let \( a_n = (-1)^n \) and \( b_n = (-1)^{n-1} \). Then \( a_n + b_n = 0 \) for all \( n \), and so \( \limsup_{n \to \infty} (a_n + b_n) = 0 \) while, \( \limsup_{n \to \infty} a_n + \limsup_{n \to \infty} b_n = 1 + 1 = 2 \).

2. Let \( \{a_n\} \) be a sequence of real numbers, and let

\[
S = \{ x \in \mathbb{R} \mid \exists \text{ a sub-sequence } a_{n_k} \text{ such that } a_{n_k} \xrightarrow{k \to \infty} x \}.
\]

(a) Show that \( L = \limsup a_n \) if and only if \( L = \sup S \).

**Solution:** Suppose \( L = \limsup a_n \). First, we claim that \( L \in S \). To see this, note that by the equivalent characterization of limsup, there exists \( n_1 \) such that

\[
a_{n_1} > L - 1.
\]
Given \( n_1 \), there exists \( n_2 > n_1 \) such that 
\[
a_{n_2} > L - \frac{1}{2}.
\]
Having chosen \( n_1 < n_2 < \cdots < n_{k-1} \), let \( n_K > n_{k-1} \) such that 
\[
a_{n_k} > L - \frac{1}{k}.
\]

**Claim.** \( a_{n_k} \xrightarrow{k \to \infty} L \).

**Proof.** Let \( \varepsilon > 0 \). Then there exists \( N \) such that for all \( n > N \),
\[
a_n < L + \varepsilon.
\]
Since \( n_k \xrightarrow{k \to \infty} \), there exists a \( K_1 \) such that for all \( k > K_1, n_k > N \). In particular, for all \( k > K_1 \),
\[
a_{n_k} < L + \varepsilon.
\]
Let \( K_2 \) such that \( 1/K_2 < \varepsilon \). Then by our choice of the subsequence \( a_{n_k} \), for all \( k > K_2 \),
\[
a_{n_k} > L - \frac{1}{k} > L - \frac{1}{K_2} > L - \varepsilon.
\]
In particular, if \( K = \max(K_1, K_2) \), and \( k > K \) then
\[
|a_{n_k} - L| < \varepsilon,
\]
and hence \( a_{n_k} \xrightarrow{k \to \infty} L \).

This shows that \( L \in S \). In particular, \( L \leq \sup S \). Suppose \( L < \sup S \). Let \( \varepsilon > 0 \) such that 
\[
L + \varepsilon < \sup S.
\]
There exists an \( N \) such that for all \( n > N \),
\[
a_n < L + \varepsilon,
\]
and so for any \( x \in S, x < L + \varepsilon \). Taking sup,
\[
\sup S \leq L + \varepsilon,
\]
a contradiction. Hence \( L = \sup S \).

(b) Formulate and prove the analogous statement for \( \lim \inf \).

**Solution:** The corresponding statement would be
\[
\liminf_{n \to \infty} a_n = \inf S.
\]
One can argue as above, or alternately, use the standard trick that if \( b_n = -a_n \), then
\[
\liminf_{n \to \infty} a_n = -\limsup_{n \to \infty} b_n.
\]

**Note.** From now on, you can use the conclusions of this exercise as a theorem. So now, you have a definition of \( \lim \sup \) and two other equivalent characterizations.
3. Find the lim sup and lim inf of the sequence \( \{a_n\} \) defined recursively by

\[ a_1 = 0, \ a_{2m} = \frac{a_{2m-1}}{2}, \ a_{2m+1} = \frac{1}{2} + a_{2m}. \]

Justify your answers with complete proofs.

**Solution:** To get some intuition, we compute the first few terms of the sequence,

\[ a_2 = 0, \ a_3 = \frac{1}{2}, \ a_4 = \frac{1}{4}, \ a_5 = \frac{3}{4}, \ a_6 = \frac{3}{8}, \ a_7 = \frac{7}{8}, \ a_8 = \frac{7}{16}, \ a_9 = \frac{15}{16}. \]

Seeing a pattern, we make the claim -

**Claim.** \( \liminf_{n \to \infty} a_n = \frac{1}{2} \) and \( \limsup_{n \to \infty} a_n = 1. \)

**Proof.** The easiest proof is to simply find a formula for the \( n \)th term. We claim that

\[ a_n = \begin{cases} 
2^{m-1}, & n = 2m + 1 \\
\frac{2^{m-1} - 1}{2^m}, & n = 2m. 
\end{cases} \]

We prove this by induction. The base cases \( n = 1 \) are seen to be true. Suppose the formula is correct for some \( n = 2m - 1 = 2(m - 1) + 1 \). We then prove the formula for \( 2m \) and \( 2m + 1 \).

\[ a_{2m} = \frac{a_{2m-1}}{2} = \frac{2^{m-1} - 1}{2m}. \]

But then again by the recursion formula,

\[ a_{2m+1} = \frac{1}{2} + a_{2m} = \frac{1}{2} + \frac{2^{m-1} - 1}{2^m} = \frac{2^m - 1}{2^m}. \]

Once we have the formula, note that \( \{a_{2m+1}\} \) is a increasing to 1 and \( \{a_{2m}\} \) is a sequence increasing to \( 1/2 \). Then clearly, \( u_N = \sup\{a_k \mid k > N\} = 1 \), and \( l_N = \inf\{a_k \mid k > N\} > \frac{2^{N-1} - 1}{2^N} \). Letting \( N \to \infty \), we complete the proof of the claim.

4. (a) Let \( \{a_n\} \) be a bounded sequence with the property that every convergent subsequence converges to the same limit \( a \). Show that the entire sequence \( \{a_n\} \) converges and \( \lim_{n \to \infty} a_n = a. \)

**Solution:** If not, then there exists an \( \varepsilon > 0 \) and a subsequence \( b_k = a_{n_k} \) such that

\[ |b_k - a| > \varepsilon \]

for all \( k \). By Bolzano-Weierstrass, since \( \{b_k\} \) is bounded, there exists a further sub-sequence \( b_{k_j} \) which converges. But \( b_{k_j} = a_{n_{k_j}} \) is also a sub-sequence of \( a_n \) and since it converges, by the hypothesis, it must converge to \( a \). But by our choice of \( \{b_k\} \), \( |b_{k_j} - a| > \varepsilon \) for all \( j \), a contradiction.

(b) Now assume that \( \{a_n\} \) is a sequence with the property that every subsequence has a further subsequence that converges to the same limit \( a \). Show that the entire sequence \( \{a_n\} \) converges and \( \lim_{n \to \infty} a_n = a. \)

**Solution:** If not, then there is an \( \varepsilon > 0 \) and a sub-sequence \( b_k = a_{n_k} \) such that \( |b_k - a| > \varepsilon \). By hypothesis, \( b_k \) has a subsequence, say \( \{b_{k_j}\} \), that converges to \( a \). But then

\[ \lim_{j \to \infty} |b_{k_j} - a| = 0, \]
which contradicts the fact that \(|b_k - a| > \varepsilon\).

5. Let \(\{a_n\}_{n=0}^{\infty}\) be a sequence of real numbers satisfying
\[
|a_{n+1} - a_n| \leq \frac{1}{2} |a_n - a_{n-1}|.
\]
Show that the sequence converges. **Hint.** Show that the sequence is Cauchy.

**Solution:** Inductively, we see that for any natural number \(k\),
\[
|a_{k+1} - a_k| \leq \frac{1}{2^k} |a_1 - a_0|.
\]
Now if \(m > n\) then by triangle inequality
\[
|a_m - a_n| = |a_m - a_{m-1} + a_{m-1} - a_{m-2} + a_{m-2} - \cdots - a_n|
\leq |a_m - a_{m-1}| + |a_{m-1} - a_{m-2}| + \cdots + |a_{n+1} - a_n|
\leq |a_1 - a_0| \sum_{k=n}^{m-1} \frac{1}{2^k}
\leq \frac{1}{2^n} |a_1 - a_0| \sum_{k=0}^{\infty} \frac{1}{2^k}
\leq 2^{-n} |a_1 - a_0|.
\]
Given \(\varepsilon > 0\), let \(N\) such that \(2^{-N} |a_1 - a_0| < \varepsilon\). Then for any \(m > n > N\), \(|a_m - a_n| < \varepsilon\), and the sequence is Cauchy.

6. Let \(S = \{n_1, n_2, \ldots\}\) denote the collection of those positive integers that do not have the digit 0 in their decimal representation. (For example \(7 \in S\) but \(101 \notin S\).) Show that \(\sum_{k=1}^{\infty} 1/n_k\) converges. **Note.** This should be a surprising result in that leaving out only a few (but of course still infinite) terms out of the harmonic series, we end up with a series that suddenly converges.

**Solution:** Consider the one-digit numbers in \(S\), namely \(\{1, 2, \ldots, 9\}\). Since each is bigger than one, the sum of reciprocals is
\[
1 + \frac{1}{2} + \cdots + \frac{1}{9} < 9.
\]
Next, consider the two-digit numbers in \(S\). There are 81 of them, and each is bigger than 10, and so the sum of reciprocals satisfies the estimate,
\[
\frac{1}{11} + \frac{1}{12} + \cdots + \frac{1}{19} + \frac{1}{21} + \cdots + \frac{1}{98} + \frac{1}{99} < \frac{81}{10}.
\]
In general, consider the subset \(S_k\) of numbers in \(S\) with \(k\) digits, that is numbers between \(10^k\) and \(10^{k+1}\). The number of such numbers is \(9^k\). That is because there are \(k\) digits, and each digit has 9 options. Moreover, all these numbers are bigger than \(10^k\), and so
\[
\sum_{n \in S_k} \frac{1}{n} < \frac{9^{k+1}}{10^k}.
\]
Summing over the reciprocals of numbers with at-most \( m \)-digits,

\[
\sum_{k=1}^m \sum_{n \in S_k} \frac{1}{n} < 9 \sum_{k=1}^m \frac{9^k}{10^k} < \frac{9}{10} \sum_{k=0}^\infty \frac{9^k}{10^k} < \frac{81}{10} \frac{1}{1 - \frac{1}{10}} = 81.
\]

In particular the partial sums of \( \sum_{k=1}^\infty 1/n_k \) are bounded by 81 and since the terms in the series are positive by the monotone convergence theorem, the series converges.

7. The Fibonacci numbers \( \{f_n\} \) are defined by

\[ f_0 = f_1 = 1, \text{ and } f_{n+1} = f_n + f_{n-1} \text{ for } n = 1, 2, \cdots. \]

For \( n = 1, 2, \cdots \), we also define \( r_n = f_{n+1}/f_n \).

(a) Find a formula for \( r_{n+1} \) in terms of \( r_n \). Dividing the above recurrence by \( f_n \), we obtain

\[ r_n = 1 + \frac{1}{r_{n-1}}, \]

or

\[ r_{n+1} = 1 + \frac{1}{r_n}. \]

(b) Show that \( f_n \geq n \) for all \( n \geq 2 \).

**Solution:** Easy proof by induction.

(c) Show that \( f_{n+1}f_{n-1} - f_n^2 = (-1)^{n+1} \).

**Solution:** We proceed by induction. For \( n = 1 \), \( f_{n+1}f_{n-1} - f_n^2 = f_2f_0 - f_1^2 = 2 \cdot 1 - 1^2 = 1 = (-1)^{1+1} \), and so the identity is verified. Suppose the identity is verified for \( n - 1 \), that is we have \( f_n f_{n-2} - f_{n-1}^2 = (-1)^n \). Then

\[
\begin{align*}
f_{n+1}f_{n-1} - f_n^2 &= (f_n + f_{n-1})f_{n-1} - f_n^2 \\
&= f_{n-1}^2 + f_n(f_{n-1} - f_n) \\
&= f_{n-1}^2 - f_n f_{n-2} = -(-1)^n = (-1)^{n+1}.
\end{align*}
\]

(d) Hence show that if \( n \geq 2 \), then

\[ |r_{n+1} - r_n| \leq \frac{1}{(n-1)^2}. \]

**Solution:** Note that

\[
|r_n - r_{n-1}| = \left| \frac{f_{n+1}}{f_n} - \frac{f_n}{f_{n-1}} \right| = \frac{1}{f_n f_{n-1}}
\]

by the identity. By part (b), \( f_n \geq n \) for all \( n \geq 2 \) and so the proof is completed by the elementary observation that \( (n-1)^2 < n(n-1) \).

(e) Hence show that the sequence of ratios \( \{r_n\} \) converge, and compute it’s limit. **Note.** This limit is the so-called golden ratio.
Solution: For any $n < m$, by the triangle inequality and part (d),

$$|r_m - r_n| \leq \frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots + \frac{1}{(m-1)^2}.$$

Since the right hand side is the tail end of a converging series, by the Cauchy criteria, for any $\varepsilon > 0$, there exists an $N$ such that for any $n, m > N$, the right hand side can be made smaller than $\varepsilon$. This shows that $\{r_n\}$ is Cauchy, and hence converges. To find the actual limit, first note that if $L = \lim_{n \to \infty} r_n$, then $L \neq 0$. Letting $n \to \infty$ on both sides of the recurrence obtained in part (a) we obtain

$$L = 1 + \frac{1}{L}.$$

Solving the quadratic $L^2 - L - 1 = 0$, we see that the roots are $(1 \pm \sqrt{5})/2$, of which the only positive root has to be $L$. 

8. Investigate the behavior of each series (convergence, divergence, conditional convergence, absolute convergence). In cases that there is a parameter ($p, q$ or $r$) find the range of values where the series exhibits the above behavior.

1. $\sum_{n=1}^\infty p^n n^p$ ($p > 0$)
2. $\sum_{n=1}^\infty (-1)^n \frac{\sqrt{n+1} - \sqrt{n}}{n^p}$
3. $\sum_{n=1}^\infty \frac{1}{p^n - q^n}$, ($0 < q < p$)
4. $\sum_{n=1}^\infty \frac{n!}{n^n}$
5. $\sum_{n=1}^\infty (\sqrt{n} - 1)^n$
6. $\sum_{n=1}^\infty \frac{1}{n^{p-r^n}}$.

Solution:

1. Let $a_n = p^n n^p$. Then $\sum_{n=1}^\infty \frac{pn^n}{n^{p+1}}$ converges absolutely for $p > 1/2$. On the other hand when $p < 0$, the series diverges. At $p = 0$, the partial sums of the series are $s_n = \sum_{k=1}^n \sqrt{k+1} - \sqrt{k} = \sqrt{n+1} - 1$ which clearly diverge. For $p \in (0, 1/2]$ the series does not converge absolutely. To check for conditional convergence we apply alternating series test.

Let $b_n = \frac{\sqrt{n+1} - \sqrt{n}}{n^p} = \frac{1}{n^p (\sqrt{n+1} + \sqrt{n})}$, and hence decreases to 0 if $p > 0$. So by the alternating series test the series converges conditionally in the range $p \in (0, 1/2]$.

2. Let $a_n = (-1)^n \frac{\sqrt{n+1} - \sqrt{n}}{n^p}$. Then

$$|a_n| = \frac{1}{n^p (\sqrt{n+1} + \sqrt{n})}.$$

Comparing this with $n^{-(p+\frac{1}{2})}$, we see that $\sum a_n$ converges absolutely for $p > 1/2$. On the other hand when $p < 0$, clearly the series diverges. At $p = 0$, the partial sums of the series are $s_n = \sum_{k=1}^n \sqrt{k+1} - \sqrt{k} = \sqrt{n+1} - 1$ which clearly diverge. For $p \in (0, 1/2]$ the series does not converge absolutely. To check for conditional convergence we apply alternating series test.

Let $b_n = \frac{\sqrt{n+1} - \sqrt{n}}{n^p} = \frac{1}{n^p (\sqrt{n+1} + \sqrt{n})}$, and hence decreases to 0 if $p > 0$. So by the alternating series test the series converges conditionally in the range $p \in (0, 1/2]$.

3. We can write

$$a_n = \frac{1}{p^n - q^n} = \frac{1}{p^n (1 - (q/p)^n)}.$$

Since $q < p$, $\lim_{n \to \infty} (q/p)^n = 0$, and so there exists an $N$ such that for all $n > N$, $(q/p)^n < 1/2$ or $(1 - (q/p)^n)^{-1} < 2$. On the other hand, for any $n$, $(1 - (q/p)^n)^{-1} < 1$, and so for $n > N$,

$$\frac{1}{p^n} < a_n < \frac{2}{p^n}.$$

By comparison test the series converges if $p > 1$ and diverges if $0 < p \leq 1$.
4. We use the ratio test. If $a_N = n^n/n!$, then
\[
\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{n^n(n+1)} = \left(1 + \frac{1}{n}\right)^n \xrightarrow{n\to\infty} e > 1,
\]
and so the series diverges.

5. We use root test. Let $a_n = (\sqrt{n} - 1)^n$. Then
\[
\sqrt[n]{a_n} = \sqrt[n]{n} - 1 \xrightarrow{n\to\infty} 0 < 1,
\]
and so the series converges.

6. Let $a_n = (1 + r^n)^{-1}$. If $|r| \leq 1$, then $\{a_n\}$ does not converge to zero, and so the series diverges. If $|r| > 1$, let $r^{-n} = 0$, and so there exists an $N \in \mathbb{N}$ such that $1 + r^{-n} > 1/2$ for all $n > N$ (note that $r$ could be negative, or else $1 + r^{-n}$ is of course bigger than 1). Then for $n > N$,
\[
|a_n| = \frac{|r|^{-n}}{1 + r^{-n}} < \frac{2}{|r|^n},
\]
and so by limit comparison test, the series is absolutely convergent for $|r| > 1$.

9. (a) Let $\{a_n\}$ be a sequence of positive real numbers. Show that
\[
\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \leq \liminf_{n\to\infty} \sqrt[n]{a_n} \leq \limsup_{n\to\infty} \sqrt[n]{a_n} \leq \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}.
\]
You may assume that each of the quantities is finite, even though the result holds true for extended reals. Hint. Proceed by contradiction. For instance, for the rightmost inequality, let $U = \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}$ and $L = \limsup_{n\to\infty} \sqrt[n]{a_n}$ and suppose $L > U$. Then use the equivalent characterizations of lim sup to draw a contradiction.

Solution: We will show that
\[
\limsup_{n\to\infty} \sqrt[n]{a_n} \leq \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}.
\]
The other inequalities also follow in a similar fashion. Denote $L = \limsup_{n\to\infty} \sqrt[n]{a_n}$ and $U = \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}$. We proceed by contradiction, so suppose $L > U$. Let $\beta \in (U, L)$. Then there exists an $N$ such that for all $n > N$,
\[
\frac{|a_{n+1}|}{a_n} < \beta,
\]
or equivalently, $|a_{n+1}| < \beta |a_n|$. Inductively, one can conclude that $|a_n| < \beta^{n-N} |a_N|$. That is, for any $n > N$,
\[
|a_n|^{1/n} < \beta^{1-N/n} |a_N|^{1/n}.
\]
Now $N$ is fixed, so taking lim sup on both sides, since $\lim_{n\to\infty} \beta^{1-N/n} |a_N|^{1/n} = \beta$, we see that
\[
\limsup_{n\to\infty} |a_n|^{1/n} \leq \beta,
\]
which is a contradiction.

(b) Show that if $\sum a_n$ converges by the ratio test, then $\sum a_n$ also converges by the root test.
Solution: If \( \sum a_n \) converges by the ratio test, then \( \limsup_{n \to \infty} |a_{n+1}/a_n| < 1 \). But then by the above set of inequalities, we necessarily have that \( \limsup_{n \to \infty} \sqrt[n]{|a_n|} < 1 \), and so the series also converges by the root test.

(c) Consider the sequence \( \{a_n\}_{n=0}^{\infty} \):

\[
a_n = \frac{1}{2^{n+(-1)^n}} = \begin{cases} \frac{1}{2^n}, & n \text{ is odd} \\ \frac{1}{2^{n+1}}, & n \text{ is even.} \end{cases}
\]

Compute (with proper justifications) \( \limsup \sqrt[n]{|a_n|} \) and \( \limsup |a_{n+1}/a_n| \). Show that the series converges by the root test. Does the ration test work?

Solution: Note that

\[
\sqrt[n]{|a_n|} = \frac{1}{2^{1+(-1)^n/n}} \xrightarrow{n \to \infty} \frac{1}{2},
\]

and so \( \limsup \sqrt[n]{|a_n|} = 1/2 \). On the other hand,

\[
\frac{a_{n+1}}{a_n} = \begin{cases} \frac{1}{2}, & n \text{ is odd} \\ 2, & n \text{ is even,} \end{cases}
\]

and so \( \limsup |a_{n+1}/a_n| = 2 \) and \( \liminf |a_{n+1}/a_n| = 1/8 \). Since \( \limsup \sqrt[n]{|a_n|} = 1/2 < 1 \), the root test says that the series \( \sum a_n \) converges. On the other hand since

\[
\liminf \frac{|a_{n+1}|}{a_n} < 1 < \limsup \frac{|a_{n+1}|}{a_n},
\]

the ratio test is inconclusive.

(d) Let \( b_n = n^n/n! \). Show that

\[
\lim_{n \to \infty} \sqrt[n]{b_n} = e.
\]

Hint. It is easier to compute the limiting ratios.

Solution: Note that

\[
\frac{a_{n+1}}{a_n} = \frac{n+1}{n} \frac{n!}{n} = \frac{n+1}{n} \frac{n!}{n!} \xrightarrow{n \to \infty} e.
\]

So \( \liminf_{n \to \infty} \frac{a_{n+1}}{a_n} = \limsup_{n \to \infty} \frac{a_{n+1}}{a_n} = e \). But then by the chain of inequalities in the first part the middle two terms are also equal, that is, \( \liminf_{n \to \infty} \sqrt[n]{a_n} = \limsup_{n \to \infty} \sqrt[n]{a_n} = e \), and so

\[
\lim_{n \to \infty} \sqrt[n]{a_n} = e.
\]

10. (a) Show that if \( a_n > 0 \), and \( \lim_{n \to \infty} na_n = l \neq 0 \), then \( \sum a_n \) diverges.

Solution: Since \( na_n \to l \neq 0 \), applying the definition of convergence with \( \varepsilon = |l|/2 > 0 \), there exists \( N \in \mathbb{N} \) such that

\[
n > N \implies |na_n - l| < \frac{|l|}{2}.
\]

In particular, for \( n > N \), \( a_n > |l|/2n \). By the comparison test, since \( \sum 1/n \) diverges, it follows that \( \sum a_n \) also diverges.
(a) For any \(n \in \mathbb{N}\)  
(b) Given that \(\sum a_n\) converges absolutely, show that \(\sum a_n^p\) also converges whenever \(p > 1\). Give a counterexample, if \(\sum a_n\) only converges conditionally.

**Solution:** Since \(\sum a_n\) converges absolutely, by the divergence test, \(|a_n| \to 0\). In particular, there exists \(N\) such that for all \(n > N\), \(|a_n| < 1\). But then for any \(p > 1\), \(|a_n|^p < |a_n|\) when \(n > N\). By comparison test, \(\sum |a_n|^p\) converges, and hence \(\sum a_n^p\) also converges. This is not true if \(\sum a_n\) only converges conditionally. For instance, consider \(a_n = (-1)^n/\sqrt{n}\) and \(p = 2\).

11. Consider each of the following propositions. Provide short proofs for those that are true and counterexamples for any that are not.

(a) If \(\sum a_n\) converges and the sequence \(\{b_n\}\) also converges, then \(\sum a_n b_n\) converges.

(b) If \(\sum a_n\) converges conditionally, then \(\sum n^2 a_n\) diverges.

**Solution:** The proposition is true. If not, then \(\sum n^2 a_n\) converges, and so \(\lim_{n \to \infty} n^2 a_n = 0\). In particular, there exists \(N\) such that for all \(n > N\), \(|a_n| < 1/n^2\), and so \(\sum a_n\) must converge absolutely by the comparison test. A contradiction!

(c) If \(\{a_n\}\) is a decreasing sequence, and \(\sum a_n\) converges, then \(\lim_{n \to \infty} n a_n = 0\).

**Solution:** The proposition is true. Since \(\sum a_n\) is convergent, \(\lim_{n \to \infty} a_n = 0\). But then since \(a_n\) is also decreasing, it follows that \(a_n \geq 0\). By the Cauchy criteria, given any \(\varepsilon > 0\), there exists an \(N\) such that for all \(n > m > N\),

\[
\sum_{k=m}^{n} a_k < \frac{\varepsilon}{2}.
\]

Applying this to \(m = \lfloor n/2 \rfloor\) with \(n > 2N\), and using the fact that \(a_n\) decreases

\[
\frac{\varepsilon}{2} > \sum_{k=\lfloor n/2 \rfloor}^{n} a_k > \frac{n a_n}{2}.
\]

So given \(\varepsilon > 0\), if \(n > 2N\), then \(|n a_n| < \varepsilon\), and hence \(\lim_{n \to \infty} n a_n = 0\).

12. (a) For any \(n \in \mathbb{N}\), show that the function \(p_n(x) = x^n\) is continuous on all of \(\mathbb{R}\). Show the explicit dependence of \(\delta\) on \(\varepsilon\) and the point that you are looking at.

**Solution:** We prove continuity at \(x = a\). Let \(\varepsilon > 0\) be given. We need to estimate

\[
|p_n(x) - p_n(a)| = |x^n - a^n| = |x - a||x^{n-1} + x^{n-2}a + \ldots + a^{n-1}|.
\]

Now, \(|x - a| < \delta\), where \(\delta > 0\) is to be chosen. Suppose, we choose \(\delta < 1\), then clearly \(|x| < |a| + 1\). A general term on the right is of the form \(x^j a^{n-1-j}\) for \(j = 0, \ldots, n-1\). So if \(\delta < 1\), we have

\[
|x^j a^{n-1-j}| < (|a| + 1)^j |a|^{n-1-j} < (|a| + 1)^{n-1}.
\]

Then by triangle inequality,

\[
|x^{n-1} + x^{n-2}a + \ldots + a^{n-1}| < n(1 + |a|)^{n-1}.
\]
So if $\delta < 1$ and $|x - a| < \delta$, then

$$|p_n(x) - p_n(a)| < n\delta(1 + |a|^{n-1}).$$

Our aim is to make this smaller than $\varepsilon$, and so simply choose

$$\delta < \frac{\varepsilon}{n(1 + |a|^{n-1})}.$$

Together with $\delta < 1$, we see that if

$$\delta = \min\left(1, \frac{\varepsilon}{n(1 + |a|^{n-1})}\right),$$

then

$$|x - a| < \delta \implies |p_n(x) - p_n(a)| < \varepsilon.$$

(b) Show that $f(x) = \sqrt{x}$ is continuous on $(0, \infty)$.

**Solution:** Let $\varepsilon > 0$, and $|x - a| < \delta$ for some $\delta > 0$ to be chosen later. Clearly

$$|\sqrt{x} - \sqrt{a}| = \frac{|x - a|}{\sqrt{x} + \sqrt{a}} < \frac{\delta}{\sqrt{x} + \sqrt{a}}.$$

Now, $a > 0$, by choosing $\delta < a/2$,

$$|x - a| < \delta \implies x > a/2.$$

So

$$\sqrt{x} + \sqrt{a} > 3\sqrt{a}/2,$$

and

$$|\sqrt{x} - \sqrt{a}| < \frac{2\delta}{3\sqrt{a}}.$$

So simply pick

$$\delta = \min\left(\frac{a}{2}, \frac{3\sqrt{a}}{2\varepsilon}\right).$$

(c) Show that $f_n(x) = x^{1/n}$ is continuous on $(0, \infty)$.

**Solution:** Again, we prove continuity at $x = a$. Let $\varepsilon > 0$. From the identity, we see that

$$x - a = (x^{1/n} - a^{1/n})(x^{1-1/n} + x^{1-2/n}a^{1/n} \ldots + a^{1-1/n}).$$

Then

$$|f_n(x) - f_n(a)| = \frac{|x - a|}{|x^{1-1/n} + x^{1-2/n}a^{1/n} \ldots + a^{1-1/n}|}.$$

Again as before, since $a > 0$, if $\delta < a/2$, then $x > a/2$ and so

$$x^{1-1/n} + x^{1-2/n}a^{1/n} \ldots + a^{1-1/n} > na^{1-1/n}c_n,$$

where $c_n$ is the constant (independent of $a$)

$$c_n = 2^{1-1/n} + 2^{1-2/n} + \ldots + 1.$$
And so if $|x - a| < \delta$ and $\delta < a/2$ we have

$$|f_n(x) - f_n(a)| < \frac{\delta}{nc_n a^{1-1/n}}.$$ 

So simply pick

$$\delta = \min\left(\frac{a}{2}, \frac{nc_n a^{1-1/n} \varepsilon}{\varepsilon}\right).$$

**Hint.** For all parts the following identity might be useful.

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}).$$