Solutions to Assignment-1

In each of the following, only use (and indicate) theorems or axioms introduced in the lectures.

1. (a) Show that $\sqrt{12}$ is irrational.

> **Solution:** If not, then $\sqrt{6} = p/q$ where p and q have no common factors. Squaring, $p^2 = 12q^2$. Since 3 divides the right-hand side, it ought to also divide the left hand side, and hence must divide p. So let $p = 3m$. Then we have $9m^2 = 12q^2$ or $3m^2 = 4q^2$, and so $3|4q^2$. But since 3 is a prime and does not divide 4, we must have that $3|q^2$ or $3|q$. But then 3 is a common factor between p and q, a contradiction!.
>
> **Note.** One cannot argue using divisibility with respect to 2 (try it!!) as in the proof of irrationality of $\sqrt{2}$.

(b) Now consider the set $E := \{\alpha \in \mathbb{Q} \mid \alpha^2 < 12\}$. Given any positive number $\beta \in \mathbb{Q}$ such that $\beta^2 < 12$, find an explicit rational number $\varepsilon > 0$ (depending of course on β), such that $(\beta + \varepsilon)^2 < 12$.

> **Solution:** We need an $\varepsilon > 0$ such that $(\beta + \varepsilon)^2 < 12$. Note that
>
> $$(\beta + \varepsilon)^2 = \beta^2 + 2\beta\varepsilon + \varepsilon^2.$$
>
> Now, since $\beta^2 < 12$, clearly, $\beta < 4$. If we then pick $\varepsilon < 1$, since $\varepsilon^2 < \varepsilon$, we automatically have $(\beta + \varepsilon)^2 < \beta^2 + 9\varepsilon$. Now, if pick ε smaller than $(12 - \beta^2)/9$, we will have $(\beta + \varepsilon)^2 < 12$. So we can take
>
> $$\varepsilon < \min\Big(1, \frac{12 - \beta^2}{9}\Big).$$

(c) Similarly, if $\beta^2 > 12$, and $\beta < 4$, find an explicit positive rational number ε such that $(\beta - \varepsilon)^2 > 12$ and yet $\beta - \varepsilon$ is an upper bound of E.

> **Solution:** We argue as above, noting that
>
> $$(\beta - \varepsilon)^2 = \beta^2 - 2\beta\varepsilon + \varepsilon^2 \geq \beta^2 - 8\varepsilon + \varepsilon^2 > \beta^2 - 8\varepsilon,$$
>
> since $\beta < 4$ and $\varepsilon^2 > 0$. Since we want this to be greater than 12, we simply choose
>
> $$\varepsilon = \frac{\beta^2 - 12}{8}.$$
>
> This solves the first part.
>
> **Claim.** With ε as above, $\beta - \varepsilon$ is also an upper bound for E. Firstly, note that $\beta - \varepsilon > 0$. So if the claim is false, then there exists a positive $\alpha \in E$ such that $\alpha \geq \beta - \varepsilon$. Squaring both sides preserves the inequality since both numbers are positive, and we obtain
>
> $$\alpha^2 \geq (\beta - \varepsilon)^2.$$
>
> This is clearly a contradiction, since the left hand side $\alpha^2 < 12$, by the definition of E, while the right hand side by construction is bigger than 12.

(d) Hence show that E has no least upper bound in $\mathbb{Q}$.

Solution: Suppose $\beta = \sup E \in \mathbb{Q}$. There are three cases.

- $\beta^2 < 12$. Then by definition $\beta \in E$. By part(b), there exists an $\varepsilon > 0$ such that $\beta+\varepsilon \in E$, and so β is not an upper bound for E. A contradiction.
- $\beta^2 > 12$. By part(c), there exists an $\varepsilon > 0$ such that $\beta - \varepsilon$ is also an upper bound for E, and so β which contradicts the fact that β is the *least* upper bound.
- $\beta^2 = 12$. By part(a), this is not possible for a rational number β.

So all three cases give contradictions, and hence E does not have a supremum in $\mathbb{Q}$.

2. Let $a, b \in \mathbb{R}$.

(a) Show that $|b| \leq a$ if and only if $-a \leq b \leq a$.

Solution: Every if and only if proof has two directions.

- $\implies$. So we assume that $|b| \leq a$. In particular this implies that $a \geq 0$. We proceed by contradiction. Suppose $b > a$. Then $b > 0$, and so $|b| = b$ contradicting the assumption that $|b| \leq a$. Similarly, if $b < -a$, then $b < 0$ and so $|b| = -b$, and so $-b \leq a$, or $b \geq -a$ which is a contradiction.
- $\impliedby$. Now we assume that $-a \leq b \leq a$. If $b \geq 0$, then $|b| = b \leq a$. If $b < 0$, then $|b| = -b$. Since $b \geq -a$, $-b \leq a$. So again $|b| \leq a$.

(b) Show that $||b| - |a|| \leq |b - a|$.

Solution: By the regular triangle inequality, $|a| \leq |a - b| + |b|$, and so $|b| - |a| \geq -|b - a|$. Again by triangle inequality, $|b| \leq |b - a| + |a|$, and so $|b| - |a| \leq |b - a|$. That is,

$$-|b - a| \leq |b| - |a| \leq |b - a|.$$

Then by part(a) above, $||b| - |a|| \leq |b - a|$.

3. Let $A, B \subset \mathbb{R}$.

(a) If $\sup A < \sup B$, then show that there is some $b \in B$ which is an upper bound for A.

Solution: Since $\sup A < \sup B$, there exists a γ such that $\sup A < \gamma < \sup B$. But then $\alpha < \gamma$ for all $\alpha \in A$. On the other hand γ cannot be an upper bound for B, and so there exists $b \in B$ such that $\gamma \leq b$. Then this b is clearly an upper bound for A.

(b) Show, by providing an example, that this is not necessarily the case if $\sup A \leq \sup B$.

Solution: Let

$$A = \{-\frac{1}{n} \mid n \in \mathbb{N}\}, \ B = [-1, 0).$$

Then $\sup A = \sup B = 0$, but no element of B is an upper bound for A.

4. Let $a < b$ be real numbers, and consider the set $T = \mathbb{Q} \cap [a, b]$. Show that $\inf T = a$ and $\sup T = b$.

Solution: We show that $\sup T = b$. Clearly b is an upper bound for T. Suppose γ is another upper bound, and suppose $\gamma < b$. Then by the density of rationals, there exists a rational $r \in \mathbb{Q}$ such that $\gamma < r < b$. Then $r \in T$ and γ cannot be an upper bound. Contradiction!

5. (a) Let $a, b \in \mathbb{R}$ such that $a \leq b + \frac{1}{n}$ for all $n \in \mathbb{N}$. Show that $a \leq b$.

Solution: If not, then $a > b$ or equivalently $a - b > 0$. Then by the corollary to the Archimedean property, there exists an integer n such that $a - b > 1/n$, contradicting the hypothesis that $a \leq b + \frac{1}{n}$ for all n. Hence we must have that $a \leq b$.

(b) Show that if $a > 0$, then there exists a natural number $n \in \mathbb{N}$ such that $\frac{1}{n} \leq a \leq n$.

Solution: By the Archimedean property, there exist natural numbers n_1 and n_2 such that

$$\frac{1}{n_1} < a \text{ and } a < n_2.$$

Now let $n = \max(n_1, n_2)$. Then $1/n \leq 1/n_1$ and $n \geq n_2$, and so

$$\frac{1}{n} \leq a \leq n.$$

(c) Let $a, b \in \mathbb{R}$ such that $a < b$. Use the denseness of $\mathbb{Q}$ to show that there are infinitely many rationals between a and b.

Solution: By the density of $\mathbb{Q}$, there is at least one rational number in (a, b). Call this r_1. Then again by density of rationals, there is at least one rational in (r_1, b); call this r_2. Having picked $r_1, r_2, \cdots, r_n$, let r_{n+1} be a rational number between (r_n, b). So we have an infinite collection of rationals $r_1, r_2, \cdots$ between a and b.

6. Let A and B be non-empty subsets of $\mathbb{R}$, and let

$$A + B := \{a + b \mid a \in A,\ b \in B\}.$$

That is, $A + B$ is the set of all sums $a + b$, where $a \in A$ and $b \in B$.

(a) Show that $\sup(A + B) = \sup A + \sup B$. **Note.** You need to separately consider the case when at least one of the two supremums on the right is ∞.

Solution: Let $\sup A = \alpha$ and $\sup B = \beta$.

- Suppose $\alpha, \beta < \infty$. Clearly $\alpha + \beta$ is an upper bound for $A + B$, and so $\sup(A + B) \leq \alpha + \beta$. Next, for any $\varepsilon > 0$, there exists an $a \in A$ and a $b \in B$ such that

$$a \geq \alpha - \frac{\varepsilon}{2},\ b \geq \beta - \frac{\varepsilon}{2}.$$

  So there exists $a + b \in A + B$ satisfying $a + b \geq \alpha + \beta - \varepsilon$. Taking supremum,

$$\sup(A + B) \geq \alpha + \beta - \varepsilon.$$

  Since this is true for all $\varepsilon > 0$, we must have $\sup(A + B) \geq \alpha + \beta$ and so $\sup(A + B) = \alpha + \beta$.

- Suppose $\alpha = \infty$. Then A is unbounded, and so must $A + B$.
- $\beta = \infty$. Same argument.

(b) $\inf(A + B) = \inf A + \inf B$.

Solution: Follows from the facts that $\inf E = -\sup(-E)$, $-(A+B) = -A+(-B)$, and part(a) above.

7. For each sequence, find the limit, and use the definition of limits to prove that the sequence does indeed converge to the proposed limit. Note that this means, given an $\varepsilon > 0$, you need to write down an N for which the definition of convergence works. Try to make the dependence of N on ε as explicit as possible.

(a) $\lim_{n\to\infty} \frac{3n+1}{6n+5}$.

Solution: **Discussion.** $(3n+1)/(6n+5) = (3+1/n)/6+5/n$. As $n \to \infty$, clearly this should tend towards $3/6$ or $1/2$. To prove that this is the limit we need to estimate

$$\left|\frac{3n+1}{6n+5} - \frac{1}{2}\right| = \frac{3}{2(6n+5)}.$$

Our aim should to be to make the right hand side smaller than a given $\varepsilon > 0$.

(b) $\lim_{n\to\infty} \left\lfloor \frac{12+4n}{3n} \right\rfloor$, where for any $x \in \mathbb{R}$, the floor function $\lfloor x \rfloor$ is the greatest integer smaller than or equal to x (for instance, $\lfloor \pi \rfloor = 3$, $\lfloor -2.3 \rfloor = -3$).

Solution: There exists an N such that for all $n > N$,

$$\frac{4}{n} < \frac{2}{3}.$$

Then for any $n > N$,

$$\left\lfloor \frac{12+4n}{3n} \right\rfloor = \left\lfloor \frac{4}{3} + \frac{4}{n} \right\rfloor = 1.$$

In particular, given any $\varepsilon > 0$ if $n > N$,

$$0 = \left| \left\lfloor \frac{12+4n}{3n} \right\rfloor - 1 \right| < \varepsilon,$$

and hence the limit exists and is 1.

8. Give an example of each of the following or state that such a request is impossible by referencing the correct theorem.

(a) Sequences $\{x_n\}$ and $\{y_n\}$ that both diverge, but $\{x_n + y_n\}$ converges.

Solution: $x_n = n$, $y_n = -n$.

(b) Sequence $\{x_n\}$ converges and $\{y_n\}$ diverges, but $\{x_n + y_n\}$ converges.

Solution: This cannot happen. If $x_n + y_n$ converge to A and x_n converges to B, then by the addition theorem for limits, $y_n = (x_n + y_n) - x_n$ converges to $A - B$.

(c) Two sequences $\{x_n\}$ and $\{y_n\}$ where $\{x_n y_n\}$ and $\{x_n\}$ converge, but $\{y_n\}$ diverges.

Solution: $x_n = 1/n$ and $y_n = (-1)^n$.

(d) Let $k \in \mathbb{N}$ be fixed. Then sequence $\{a_n\}_{n=1}^{\infty}$ converges but $\{a_{n+k}\}_{n=1}^{\infty}$ might not converge, or even if it converges, might not converge to the same limit.

> **Solution:** Suppose $\lim_{n\to\infty} a_n = L$.
> **Claim.** If k is fixed, then $a_{n+k} \to L$ as $n \to \infty$.
> **Proof.** Since $a_n \to L$, given any $\varepsilon > 0$, there exists N such that $n > N$ implies $|a_n - L| < \varepsilon$. But then
> $$n > N \implies |a_{n+k} - L| < \varepsilon,$$
> since $n + k > N$.

9. Let $\{x_n\}$ and $\{y_n\}$ be two sequences, and let $\{z_n\}$ by the "shuffled" sequence $\{x_1, y_1, x_2, y_2, \cdots\}$.

(a) Find a general formula for z_n.

> **Solution:**
> $$z_n = \begin{cases} x_m, & n = 2m - 1 \\ y_m, & n = 2m. \end{cases}$$

(b) Show that $\{z_n\}$ converges if and only if **both** $\{x_n\}$ and $\{y_n\}$ converge to the same value.

> **Solution:**
>
> - $\implies$. Suppose z_n converges to L. Let $\varepsilon > 0$. Then there exists an N such that for all $n > N$,
> $$|z_n - L| < \varepsilon.$$
> If $n = 2m - 1$ or $n = 2m$, then $n > N$ implies that $m > (N + 1)/2$. Then from part(a), for all $m > (N + 1)/2$,
> $$|x_m - L| < \varepsilon, \ |y_m - L| < \varepsilon,$$
> and so
> $$\lim_{n\to} x_n = \lim_{n\to\infty} y_n = L.$$
> - $\impliedby$ Suppose $\lim_{n\to} x_n = \lim_{n\to\infty} y_n = L$. Let $\varepsilon > 0$. Then there exists an N_1 such that for all $m > N_1$, $|x_m - L| < \varepsilon$. Similarly, there exists an N_2 such that for all $m > N_2$, $|y_m - L| < \varepsilon$. Let
> $$N = \max(2N_1 - 1, 2N_2).$$

10. Let $a_1 = 1$ and
$$a_{n+1} = \frac{1}{3}(a_n + 1)$$
for $n > 1$.

(a) Find a_2, a_3 and a_4.

> **Solution:** $a_2 = 2/3, \ a_3 = 5/9, \ a_4 = 14/27.$

(b) Use induction to show that $a_n > 1/2$ for all n.

Solution: For the base case, $n = 1$, clearly $a_1 = 1 > 1/2$. Suppose $a_n > 1/2$. Then

$$a_{n+1} = \frac{a_n + 1}{3} > \frac{3/2}{3} = \frac{1}{2}.$$

(c) Show that $\{a_n\}$ is a convergent sequence and compute it's limit.

Solution: From part(a) it seems that the sequence is decreasing. Let us try to prove that. Since $a_n > 1/2$, $1 < 2a_n$ and so, we have

$$a_{n+1} = \frac{a_n + 1}{3} < \frac{a_n + 2a_n}{3} = a_n.$$

So the sequence is decreasing and bounded below, hence by the monotone convergence theorem, the sequence converges. Suppose $\lim_{n\to\infty} a_n = L$, then taking limits as $n \to \infty$ for the recurrence,

$$L = \frac{L+1}{3},$$

or $L = 1/2$.

11. (a) Show that the sequence

$$\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \cdots$$

converges, and find it's limit.

Solution: Let $a_1 = \sqrt{2}$ and

$$a_{n+1} = \sqrt{2 + a_n}.$$

Then $\{a_n\}$ is precisely the sequence we are considering.

- We first claim that $0 < a_n < 2$ for all n. We prove this by induction. For $n = 1$, $a_1 = \sqrt{2} \approx 1.414 < 2$ and is positive. Suppose $a_n < 2$, then

  $$a_{n+1} = \sqrt{2 + a_n} < \sqrt{2+2} = 2,$$

  and so $a_{n+1} < 2$. Positivity is also clear since we are taking the positive square root. This completes the inductive step and hence the proof.

- Next, we claim that a_n is an increasing sequence. To see this, note that since $a_n < 2$, we have $a_n + 2 > 2a_n$, and so

  $$a_{n+1} = \sqrt{2 + a_n} > \sqrt{2a_n} > \sqrt{a_n^2} = a_n.$$

  So $a_{n+1} > a_n$ for all n and the sequence is increasing.

Then by the theorem on convergence of monotonic sequences, $\{a_n\}$ converges. Suppse $\lim_{n\to\infty} a_n = L$. Then taking limits as $n \to \infty$ on both sides of $a_{n+1} = \sqrt{2 + a_n}$, we see that

$$L = \sqrt{2 + L},$$

or $L = 1$ or 2. But since $\{a_n\}$ is always increasing, and $a_1 > 1$, L has to be 2. So

$$\lim_{n\to\infty} a_n = 2.$$

(b) Does the sequence $\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}$ converge? Give a complete proof. If it does converge, also find the limit.

Solution: The sequence is now given by the recurrence, $x_1 = \sqrt{2}$, and $x_{n+1} = \sqrt{2x_n}$.

- We again claim that $x_n < 2$ for all n. Again, we proceed by induction. Clearly $x_1 < 2$ which is the base case. For the inductive step, suppose $x_n < 2$. Then

$$x_{n+1} = \sqrt{2x_n} < \sqrt{2 \cdot 2} = 2.$$

- Next we claim that x_n is increasing. This follows from

$$x_{n+1} = \sqrt{2x_n} > \sqrt{x_n^2} = x_n.$$

By the theorem on convergence of monotonic sequences, the sequence must converge. We compute the limit as before

$$L = \lim_{n\to} x_{n+1} = \lim_{n\to\infty} \sqrt{2x_n} = \sqrt{2L},$$

and so $L = 2$.

12. (Arithmetic and geometric means)

(a) Show that

$$\frac{x+y}{2} \geq \sqrt{xy}.$$

The quantity on the left of center is the arithmetic mean, and the quantity on the right of center is the geometric mean.

Solution: We have the following sequence of inequalities

$$\begin{aligned} \frac{x+y}{2} &\geq \sqrt{xy} \\ \iff x + y - 2\sqrt{xy} &\geq 0 \\ \iff (\sqrt{x} - \sqrt{y})^2 &\geq 0. \end{aligned}$$

Since the final inequality is clearly true, and since all implications are reversible (note the $\iff$), the first inequality also has to be true.

(b) Now, let $0 \leq x_1 \leq y_1$, and define recursively,

$$x_{n+1} = \sqrt{x_n y_n}, \; y_{n+1} = \frac{x_n + y_n}{2}.$$

Show that both $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} y_n$ exist and are equal.

Solution:

- We first claim that $x_n \leq x_{n+1} \leq y_{n+1} \leq y_n$ for all n. To see this, note first that $x_n \leq y_n$ for all n by part(a), since x_n is the geometric mean and a y_n is the arithmetic mean. Then $x_{n+1} = \sqrt{x_n y_n} \geq \sqrt{x_n^2} = x_n$. Similarly we can show that $y_n \geq y_{n+1}$.

- From the above claim it follows that $x_1 \leq x_n \leq y_n \leq y_1$ and that $\{x_n\}$ is an increasing sequence and $\{y_n\}$ is a decreasing sequence. Hence by the theorem on monotone convergence, both sequences converge. Let

  $$\lim_{n\to\infty} x_n = X, \quad \lim_{n\to\infty} y_n = Y,$$

  Then

  $$X = \sqrt{XY} \text{ and } Y = \frac{X+Y}{2}.$$

  From the second equation, it follows that $X = Y$.