Solutions to Assignment-0

(not to be handed in)

1. (De Morgan's laws) Let $\{A_\alpha\}_{\alpha\in I}$ be a collection of subsets of a larger set $\mathcal{B}$. I is simply an indexing set, that could be finite or infinite.

   (a) Show that
$$\Big(\cup_{\alpha\in I} A_\alpha\Big)^c = \cap_{\alpha\in I} A_\alpha^c,$$
   where for any subset A, $A^c = \mathcal{B}\setminus A$ is the complement.

   Solution: Discussion. The standard way of showing that two sets A and B are equal, i.e $A = B$, is to show that $A \subseteq B$ and $B \subseteq A$. To show $A \subseteq B$, the usual method is to start with an arbitrary element $x \in A$, and to show that in fact $x \in B$.

   Formal proof. We show the two inclusions.

   - Let $x \in \Big(\cup_{\alpha\in I} A_\alpha\Big)^c$. Then $x \notin \cup_{\alpha\in I} A_\alpha$, and by the definition of union, this means that $x \notin A_\alpha$ for all $\alpha \in I$. That is, $x \in A_\alpha^c$ for all $\alpha \in I$. But then by the definition of intersection, this means that $x \in \cap_{\alpha\in I} A_\alpha^c$. Since x was arbitrary, this shows that
$$\Big(\cup_{\alpha\in I} A_\alpha\Big)^c \subseteq \cap_{\alpha\in I} A_\alpha^c.$$
   - For the reverse inclusion, let $x \in \cap_{\alpha\in I} A_\alpha^c$ be an arbitrary element. Then $x \in A_\alpha^c$ for all $\alpha \in I$, and so $x \notin A_\alpha$ for all $\alpha \in I$. Then by the definition of unions, this implies that $x \notin \cup_{\alpha\in I} A_\alpha$, and so
$$\cap_{\alpha\in I} A_\alpha^c \subseteq \Big(\cup_{\alpha\in I} A_\alpha\Big)^c.$$

   (b) Show that
$$\Big(\cap_{\alpha\in I} A_\alpha\Big)^c = \cup_{\alpha\in I} A_\alpha^c.$$

   Solution: The proof is almost exactly the same as above. Using the if and only if symbol ($\iff$) we can actually combine the two steps.
$$\begin{aligned} x \in \Big(\cap_{\alpha\in I} A_\alpha\Big)^c &\iff x \notin \cap_{\alpha\in I} A_\alpha, \\ &\iff x \notin A_\alpha \text{ for some } \alpha \in I, \\ &\iff x \in A_\alpha^c \text{ for some } \alpha \in I, \\ &\iff x \in \cup_{\alpha\in I} A_\alpha^c, \end{aligned}$$
   and hence
$$\Big(\cap_{\alpha\in I} A_\alpha\Big)^c = \cup_{\alpha\in I} A_\alpha^c.$$

2. Decide which of the following statements are true and give a complete proof. For statements that are false provide a counter example.

   (a) If $A_1 \supseteq A_2 \cdots$ are all sets containing an infinite number of elements, then $\cap_{n=1}^{\infty} A_n$ is also an infinite set.

   Solution: No. In fact the intersection could even be empty. Let

   $$A_n = \mathbb{N} \setminus \{1, 2, 3, \cdots, n\}.$$

   Then $A_1 \supset A_2 \cdots$ and each set is infinite, but $\cap_{n=1}^{\infty} A_n$ is empty, since for any integer m, $m \notin A_n$ for $n > m$, so cannot be in the common intersection.

   (b) If $A_1 \supseteq A_2 \cdots$ are all finite non-empty sets of real numbers, then $\cap_{n=1}^{\infty} A_n$ is also finite and non-empty.

   Solution: This is true. Firstly, $\cap_{n=1}^{\infty} A_n \subset A_1$ and hence is finite since A_1 is finite. To show that the intersection is non-empty, we in fact prove the following stronger statement.

   Claim. There exists an $N \in \mathbb{N}$ such that $A_n = A_N$ for all $n \geq N$. That is, after a certain point, all sets have to be the identical.

   Proof. If not, then for all $N \in \mathbb{N}$, there is an integer $n > N$ such that $A_n \subsetneq A_N$. The basic idea is that if a set is a strict subset, it has strictly fewer elements. But since all sets are nested inside A_1, A_1 is a finite set, and all sets are non-empty, this process cannot go on for ever.

   More formally, suppose A_1 has m elements. Then there is some n_1 such that $A_{n_1} \subsetneq A_1$. Let A_{n_1} have m_1 elements. Then $m_1 < m$. Now again, there is some n_2 such that $A_{n_2} \subsetneq A_{n_1}$, and if A_{n_2} has m_2 elements, then $m_2 < m_1 < m$. We can continue this process indefinitely, to construct a sequence of subsets A_{n_k} with number of elements m_k such that $m_k < m_{k-1}$. Because of the strict inequality, eventually $m_k = 0$ for some k, contradicting the fact that all subsets are non-empty.

3. Use induction to prove the following.

   (a) $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

   Solution:

   - **Base Case.** $n = 1$. Clearly both L.H.S and R.H.S are 1.
   - **Inductive step.** Suppose the identity is true for some n. We want to prove it for $n + 1$.

   $$\begin{aligned} 1 + 2^2 + \cdots + n^2 + (n+1)^2 &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \qquad \text{(use identity for } n\text{)}, \\ &= (n+1)\Big[\frac{n(2n+1)}{6} + n + 1\Big], \\ &= (n+1)\Big[\frac{2n^2+7n+6}{6}\Big] \\ &= \frac{(n+1)(n+2)(2n+3)}{6} \\ &= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}, \end{aligned}$$

   and hence the identity also holds for $n + 1$.

(b) $1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$. **Note.** This is rather beautiful, that the sum of first n cubes is equal to the square of the sum of the first n numbers.

Solution:

- **Base Case.** $n = 1$, both sides are 1.
- **Inductive step.** Suppose identity is true for some n. Then,

$$\begin{aligned} 1^3 + 2^3 + \cdots + n^3 + (n+1)^3 &= \left[\frac{n(n+1)}{2}\right]^2 + (n+1)^3 \\ &= (n+1)^2\left[\frac{n^2}{4} + n + 1\right] \\ &= (n+1)^2 \frac{n^2 + 4n + 4}{4} \\ &= \frac{(n+1)^2(n+2)^2}{4} \\ &= \left[\frac{(n+1)((n+1)+1)}{2}\right]^2. \end{aligned}$$

(c) $11^n - 4^n$ is always divisible by 7.

Solution:

- **Base Case.** $n = 1$. Then $11^n - 4^n = 7$ which is clearly divisible by 7.
- **Inductive step.** Suppose the statement is true for n. Now, noting that $11 = 7 + 4$,

$$\begin{aligned} 11^{n+1} - 4^{n+1} &= 11 \cdot 11^n \cdot -4 \cdot 4^n \\ &= 7 \cdot 11^n + 4(11^n - 4^n). \end{aligned}$$

The first term on the right is clearly divisible by 7. The second term on the right has a factor of $11^n - 4^n$ which by inductive hypothesis is also divisible by 7, and so $11^{n+1} - 4^{n+1}$ is also divisble by 7, thereby completing the inductive step, and hence the entire proof.

(d) $\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$.

Solution:

- **Base Case.** $n = 1$, This is trivial since $1 = 1$.
- **Inductive step.** Suppose the inequality is prove for n. Then

$$\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n} + \frac{1}{(n+1)^2}$$

by the inductive hypothesis. We want the the terms on the right involving n to be smaller than $-1/(n+1)$. To this effect note that

$$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} \geq \frac{1}{(n+1)^2},$$

and so

$$2 - \frac{1}{n} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n+1},$$

completing the proof of the inductive step.

4. Consider the inequality

$$\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \leq 2.$$

By part(d) above, this is clearly true. But now try to prove the inequality directly by induction. Why does it not work?

Solution: Again the base case is trivial since $1 < 2$. For the inductive step suppose for some n,

$$\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \leq 2$$

. Then for $(n+1)$,

$$\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2} \leq 2 + \frac{1}{(n+1)^2},$$

which is clearly not smaller than 2. So the inductive step cannot be completed. The problem is that if one tries to prove a weaker statement using induction, in the inductive step, one has a weaker statement to exploit, and hence the inductive step might fail.