MATH 104 : Mid-Term

16 July, 2018

Name: ______________________________

- You have 100 minutes to answer the questions.
- Use of calculators or study materials including textbooks, notes etc. is not permitted.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.
- For questions with multiple parts, you can solve a part assuming the previous parts and get full credit for that particular part.

Question	Points	Score
1	18	
2	16	
3	12	
4	10	
5	14	
Total:	70	

1. (18 points) For each of the following, either give an example, or state that the request is impossible. If a request is impossible, provide a brief but compelling argument. For any example given, you do not need to prove that it has the required property.

   Unsolicited advice. Think each of these through carefully. Even if an answer pops out immediately, there is no harm in being careful.

   (a) A function f that is discontinuous at some p, but $\lim_{h\to 0}[f(p+h) - f(p-h)] = 0$.

   > **Solution:** Let
   > $$f(x) = \begin{cases} 0, & x \neq 0 \\ 1, & x = 0, \end{cases}$$
   > and take $p = 0$.

   (b) An infinite bounded set $S \subset \mathbb{R}$ such that $\sup S$ is **<u>not</u>** a limit point of S.

   > **Solution:** $S = [0,1] \cup \{2\}$. Then $\sup S = 2$, but 2 is an isolated point of the set.

   (c) A continuous, non-constant function $f : [a,b] \to \mathbb{R}$ such that the range $f([a,b])$ consists of only irrational numbers.

   > **Solution:** Impossible. By the intermediate value theorem, if f is non-constant, then $f([a,b])$ is also an interval, and hence must contain rational numbers.

(d) A continuous function $f : [a, b] \to \mathbb{R}$ such that $f(t) > 0$ for all t, but $1/f$ is unbounded on $[a, b]$.

Solution: Impossible. By the extremum value theorem, the minimum of f is attained. That is, there is some p such that $f(t) \geq f(p) > 0$ for some $p \in [a, b]$. Since both $f(t)$ and $f(p)$ are positive, this implies that $1/f$ is bounded above by $1/f(p)$ and below by zero.

(e) A sequence $\{a_n\}$ with $\limsup_{n\to\infty} a_n = 1$ such that $a_n < 1$ for all n, and $a_n = 0$ for an infinite number of indices n.

Solution: Let

$$a_n = \begin{cases} 1 - \frac{1}{n}, & n \text{ is odd} \\ 0, & n \text{ is even.} \end{cases}$$

(f) A sequence $\{a_n\}_{n=1}^{\infty}$ where $0 \leq a_n \leq 1/n$ for all n, but $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ diverges.

Solution: Let

$$a_n = \begin{cases} \frac{1}{n+1}, & n \text{ is odd} \\ 0, & n \text{ is even.} \end{cases}$$

Then the even partial terms satisfy

$$\begin{aligned} s_{2m} &= \sum_{k=1}^{2m}(-1)^{k-1}a_k \\ &= \frac{1}{2}\sum_{j=1}^{m}\frac{1}{j} \xrightarrow{m\to\infty} \infty. \end{aligned}$$

So the subsequence $\{s_{2m}\}$ diverges, and hence the series cannot converge.

2. (a) (6 points) Given an $\varepsilon > 0$, find a $\delta > 0$ such that for all $x \in (1-\delta, 1+\delta)$,

$$\left|\frac{1}{2} - \frac{x}{1+x^2}\right| < \varepsilon.$$

Solution: We have

$$\left|\frac{1}{2} - \frac{x}{1+x^2}\right| = \frac{|x-1|^2}{2(1+x^2)} \leq \frac{|x-1|^2}{2}.$$

So simply take $\delta = \sqrt{2\varepsilon}$. Then $|x-1| < \delta$ implies

$$\left|\frac{1}{2} - \frac{x}{1+x^2}\right| < \varepsilon.$$

(b) (7 points) Let $f(x) = |x|^3$. Show that f is differentiable on all of $\mathbb{R}$, and that $f'(x) = 3x|x|$.

Solution: Now that

$$f(x) = \begin{cases} x^3, \ x \geq 0 \\ -x^3, \ x < 0. \end{cases}$$

So when $x \neq 0$, f is clearly differentiable at x and moreover, $f'(x) = 3x^2$ if $x > 0$ and $-3x^2$ when $x < 0$. At zero, we see that the difference quotient is

$$\varphi(x) = \frac{f(x) - f(0)}{x - 0} = \frac{|x|^3}{x} = x^2 \xrightarrow{x \to 0} 0.$$

So f is differentiable at $x = 0$ and $f'(0) = 0$. So

$$f'(x) = \begin{cases} 3x^2, \ x \geq 0 \\ -3x^2, \ x < 0, \end{cases}$$

or simply $f'(x) = 3x|x|$.

(c) (3 points) Does $f''(0)$ exist? If so, compute it's value. If not, give a proper justification.

Solution: We once again compute the difference quotient.

$$\varphi(x) = \frac{f'(x) - f'(0)}{x} = 3|x| \xrightarrow{x \to 0} 0.$$

So f'' exists at $x = 0$ and $f''(0) = 0$.

3. Let $f : (-1, 1) \to \mathbb{R}$ be defined by

$$f(x) = \frac{1}{\sqrt{1-x}}.$$

(a) (3 points) Write down the formulae for $f'(x)$, $f''(x)$ and $f^{(3)}(x)$.

Solution: Note that $f(x) = (1-x)^{-\frac{1}{2}}$. Using power rule and chain rule,

$$f'(x) = \frac{1}{2}(1-x)^{-\frac{3}{2}}, \ f'(x) = \frac{3}{4}(1-x)^{-\frac{5}{2}}, \ f'(x) = \frac{15}{8}(1-x)^{-\frac{7}{2}}.$$

(b) (6 points) Find a degree **two** polynomial $p(x) = ax^2 + bx + c$, such that for all $x \in [-1/2, 1/2]$,

$$|f(x) - p(x)| \leq \frac{5}{\sqrt{2}}|x|^3.$$

Solution: Let

$$p(x) = T_2(0, x) = 1 + \frac{1}{2}x + \frac{3}{4}x^2.$$

Then by Taylor's theorem, for all $x \in [-1/2, 1/2]$, there exists a c (depending on x) between 0 and x such that

$$f(x) - p(x) = \frac{f^{(3)}(c)}{6}x^3 = \frac{5}{16}(1-c)^{-\frac{7}{2}}x^3.$$

Since $c \in [-1/2, 1/2]$, $1 - c > 1/2$ and so

$$|f(x) - p(x)| \leq \frac{5}{16}2^{\frac{7}{2}}|x|^3 = \frac{5}{\sqrt{2}}|x|^3.$$

(please turn over for additional space to answer this part)

(c) (3 points) For the polynomial found in part (i) above, calculate

$$\lim_{x\to 0} \frac{\frac{1}{\sqrt{1-x}} - p(x)}{x^3},$$

if it exists (no justification needed), or prove that the limit does not exist.

Solution: Using L'Hospital's rule or Taylor's series, one can check that the limit is simply the coefficient of x^3 in the Taylor series, namely

$$\lim_{x\to 0} \frac{\frac{1}{\sqrt{1-x}} - p(x)}{x^3} = \frac{15}{8 \times 3!} = \frac{5}{16}.$$

4. (a) (5 points) Let $\alpha > 1$. Prove that for all $y > x > 1$,

$$\frac{y^\alpha - x^\alpha}{y - x} \geq \alpha x^{\alpha-1}.$$

Solution: Let $f(x) = x^\alpha$. Then $f'(x) = \alpha x^{\alpha-1}$. By the mean value theorem, there exists $c \in [x, y]$ such that

$$\frac{y^\alpha - x^\alpha}{y - x} = \alpha c^{\alpha-1} \geq \alpha x^{\alpha-1},$$

since $c \geq x$ and $\alpha - 1 > 0$.

(b) (5 points) Use the above part with $\alpha = 3/2$, to show that $f(x) = x\sqrt{x}$ is not uniformly continuous on $[1, \infty)$.

Solution: From the above part, we get the inequality

$$y\sqrt{y} - x\sqrt{x} \geq \frac{3}{2}\sqrt{x}(y - x).$$

Consider the sequences,

$$x_n = n^2, \; y_n = n^2 + \frac{1}{2n}.$$

Then $|y_n - x_n| < 1/n$ and yet

$$|f(y_n) - f(x_n)| = f(y_n) - f(x_n) \geq \frac{3}{4}.$$

So we have found a contradiction to the definition of uniform continuity for $\varepsilon = 3/4$.

5. This final problem is about an algorithm to compute square roots. Let $x_1 > \sqrt{3}$ and define $x_2, x_3, \cdots$ recursively by

$$x_{n+1} = \frac{1}{2}\Big(x_n + \frac{3}{x_n}\Big)$$

(a) (2 points) Prove that $x_n > \sqrt{3}$ for all n.

Solution: Using the recurrence,

$$x_n^2 - 3 = \frac{x_{n-1}^2 + 6 + \frac{9}{x_{n-1}^2}}{4} - 3 = \frac{x_{n-1}^2 - 6 + \frac{9}{x_{n-1}^2}}{4} = \frac{1}{4}\Big(x_{n-1} - \frac{3}{x_{n-1}}\Big)^2 \geq 0$$

Hence $x_n^2 \geq 3$. So either $x_n > \sqrt{3}$ or $x_n < -\sqrt{3}$. But by induction it is easy to see that $x_n > 0$ for all n, and so $x_n > \sqrt{3}$.

(b) (3 points) Prove that $\{x_n\}$ is a decreasing sequence.

Solution: Since $x_n \geq \sqrt{3}$,

$$x_{n+1} = \frac{1}{2}\Big(x_n + \frac{3}{x_n}\Big) \leq \frac{1}{2}\Big(x_n + \frac{x_n^2}{x_n}\Big) = x_n.$$

(c) (3 points) Prove that $\{x_n\}$ is convergent (quote the relevant theorem), and that $\lim_{n\to\infty} x_n = \sqrt{3}$.

Solution: Convergence follows from the monotone convergence theorem and part(a) and (b) above. Taking limits on both sides of the recurrence, if $\lim_{n\to\infty} x_n = L$, then

$$L = \frac{1}{2}\Big(L + \frac{3}{L}\Big),$$

which can be easily solved to give $L = \pm\sqrt{3}$. Again, since $\sqrt{x_n} > \sqrt{3} > 0$ for all n, this shows that $L = \sqrt{3}$.

Now, let $\varepsilon_n = x_n - \sqrt{3}$, that is, ε_n is the error in the approximation of $\sqrt{2}$ by x_n.

(d) (3 points) Show that

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{3}}.$$

Solution:

$$\begin{aligned}
\varepsilon_{n+1} &= \frac{1}{2}\Big(x_n + \frac{3}{x_n}\Big) - \sqrt{3} \\
&= \frac{x_n^2 + 3 - 2x_n\sqrt{3}}{2x_n} \\
&= \frac{(x_n - \sqrt{3})^2}{2x_n} \qquad = \frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{3}}.
\end{aligned}$$

The final inequality follows from part(a) since $x_n > \sqrt{3}$.

(e) (3 points) Hence show that if $\beta = 2\sqrt{3}$, then

$$\varepsilon_{n+1} < \beta\Big(\frac{\varepsilon_1}{\beta}\Big)^{2^n}.$$

Solution: We prove this by induction.

- **Base case $n = 1$.** Then by part(d) above,

$$\varepsilon_2 < \frac{\varepsilon_1^2}{2\sqrt{3}} = \beta\Big(\frac{\varepsilon_1}{\beta}\Big)^2.$$

- **Inductive step.** Now suppose the estimate is proved for $1, \cdots, n-1$. Then

$$\begin{aligned}
\varepsilon_n &< \frac{\varepsilon_{n-1}^2}{2\sqrt{3}} && \text{(by part(d))} \\
&< \frac{1}{\beta}\Big(\beta\Big(\frac{\varepsilon_1}{\beta}\Big)^{2^{n-1}}\Big)^2 && \text{(by the inductive hypothesis)} \\
&= \beta\Big(\frac{\varepsilon_1}{\beta}\Big)^{2^n}.
\end{aligned}$$

Remark. This shows that the algorithm is fantastically fast. For instance, if $x_1 = 2$ then already $\varepsilon_5 < 4 \cdot 10^{-16}$, so that the answer is correct up to 14 decimal places by just the fifth iteration of the algorithm!