Practice problems for the final

1. Show the following using just the definitions (and no theorems).
   (a) The function
   $$f(x) = \begin{cases} x^2, & x \geq 0 \\ 0, & x < 0, \end{cases}$$
   is differentiable on $\mathbb{R}$, while $f'(x)$ is not differentiable at 0.
   (b) The function $f(x) = x^3 + x$ is continuous but not uniformly continuous on $\mathbb{R}$.
   (c) The sequence of functions
   $$f_n(x) = \frac{nx}{n+1}$$
   converges point wise on $\mathbb{R}$, uniformly on bounded intervals (a, b), but does NOT converge uniformly on $\mathbb{R}$.
2. Let $f : (0, 1) \to \mathbb{R}$ be a differentiable function such that $|f'(t)| \leq 1$ for all t. Show that the sequence $a_n = f(1/n)$ converges.
3. Let
   $$f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0, & \text{otherwise.} \end{cases}$$
   (a) Compute the upper and lower integrals on $[0, 1]$.
   (b) Now do the same for the interval $[-1, 1]$.
4. Suppose f is a continuous real valued function on $[0, \infty)$ which is differentiable on $(0, \infty)$ and satisfies
   $$f'(t) > f(t)$$
   for all $t \in (0, \infty)$. If $f(0) = 1$, show that $f(t) > e^t$ for all t.
5. For any $\vec{x} = (x_1, x_2)$ and $\vec{y} = (y_1, y_2)$ in $\mathbb{R}^2$, define
   $$d(\vec{x}, \vec{y}) = \max(|x_1 - y_1|, |x_2 - y_2|).$$
   (a) Show that $(\mathbb{R}^2, d)$ is a metric space.
   (b) Show that
   $$\frac{|\vec{x} - \vec{y}|}{\sqrt{2}} \leq d(\vec{x}, \vec{y}) \leq |\vec{x} - \vec{y}|.$$
   (c) Hence, show that $(\mathbb{R}^2, d)$ is a complete metric space.
   (d) Also show that a subset K is compact in $(\mathbb{R}^2, d)$ if and only if it is closed and bounded.
6. Consider the rectangle $R \subset \mathbb{R}^2$ formed by the edges $x = \pm 2$, $y = 0$ and $y = 1$ (here we consider only the boundary rectangle, and not the interior). We can think of R as a metric space with the metric induced from $\mathbb{R}^2$.
   (a) Describe the sets $B_1(\vec{0})$ and $B_{\leq 1}(\vec{0})$.

(b) Show that $\overline{B_1(\vec{0})} \neq B_{\leq 1}(\vec{0})$.

(c) On the other hand, show that for any metric space (X,d),

$$\overline{B_r(p)} \subseteq B_{\leq r}(p).$$

7. Let (X,d) be a metric space. As usual, denote by $\overline{S}$ and $\text{int}(S)$, the closure and interior of a subset S respectively. Let $\mathcal{F}$ be an arbitrary collection of subsets.

   (a) Show that

   $$\cup_{A\in\mathcal{F}}\text{int}(A) \subseteq \text{int}(\cup_{A\in\mathcal{F}}A)$$

   (b) Give an example of a metric space (X,d) and a finite collection of subsets $\mathcal{F}$ for which equality does not hold.

   (c) Show that

   $$\cup_{A\in\mathcal{F}}\overline{A} \subseteq \overline{\cup_{A\in\mathcal{F}}A}.$$

   (d) Give an example of a metric space (X,d) and an infinite family of subsets $\mathcal{F}$ for which equality does not hold above.

   (e) Show that if $\mathcal{F}$ is a *finite* collection of sets, then

   $$\cup_{A\in\mathcal{F}}\overline{A} = \overline{\cup_{A\in\mathcal{F}}A}.$$

8. A point $p \in X$ is called a *fixed point* of a map $f : X \to X$ if $f(p) = p$.

   (a) If (X,d) is a compact metric space, and f satisfies

   $$d(f(x), f(y)) < d(x,y),$$

   for $x \neq y$, show that f has a fixed point in X and that fixed point is unique. **Hint.** Consider the minimum of $d(x, f(x))$.

   (b) Show that the statement is no longer true if X is merely assumed to be complete, by considering the following example - $f : (-\infty, \infty) \to \mathbb{R}$ given by

   $$f(t) = t + \frac{1}{1+e^t}.$$

   Hint. Show that $0 < f'(t) < 1$ for all t.

   (c) If X is complete, $f : X \to X$, and

   $$d(f(x), f(y)) \leq \alpha d(x,y)$$

   for some $\alpha < 1$, then show that there is a unique fixed point in X. **Hint.** Let $x_0 \in X$ be any point, and define $x_{n+1} = f(x_n)$. , and show that if $m > n$,

   $$d(x_m, x_n) \leq \frac{\alpha^n d(x_1, x_0)}{1-\alpha}.$$

9. Let K be a compact subset of a metric space (X,d) and F a closed subset.

   (a) Show that $K \cap F$ is a compact subset.

   (b) If $K \cap F = \phi$, show that

   $$\inf_{x\in K,\ y\in F} d(x,y) > 0.$$

   (c) Providing an example, argue that if K is assumed to be only closed, then the infimum could be zero.

10. Let (X,d) be a metric space.

(a) If $p_a, \cdots, p_n$ be a finite collection of points, show using only the definition that $X \setminus \{p_1, \cdots, p_n\}$ is an open set.

(b) GIve an example of a metric space and a countable collection of points $\{p_k\}$ such that $X \setminus \{p_k\}_{k=1}^{\infty}$ is dense, infinite but not open.

11. Define a sequence of functions on $[0, 2]$ by

$$f_n(x) = \sqrt{\frac{x^2 + n}{x + n}}.$$

(a) Show that the sequence converges uniformly to some continuous function f on $[0, 2]$.

(b) Compute

$$\lim_{n \to \infty} \int_0^2 f_n(t)\, dt,$$

and justify your answer.

12. The Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Clearly the function is well defined and finite on $(1, \infty)$.

(a) Show that the series converges uniformly on $[\alpha, \infty)$ for all $\alpha > 1$.

(b) Show that $\zeta(s)$ is differentiable on $(1, \infty)$ with

$$\zeta'(s) = -\sum_{n=1}^{\infty} \frac{\ln(n)}{n^s}.$$