Assignment-7

(Due 07/30)

Please hand in all the 8 questions in red

1. Consider the sequence of functions $f_n : [0,1] \to \mathbb{R}$ defined by

$$f_n(x) = \frac{x^2}{x^2 + (1-nx)^2}.$$

   (a) Show that the sequence of functions converges pointwise as $n \to \infty$, and compute the limit function $f(x)$.

   (b) Show that the sequence is not equicontinuous on $[0,1]$.

   (c) Which theorem in the notes implies that f_n does not converge uniformly to f on $[0,1]$?

   (d) Show that $f_n \xrightarrow{u.c} f$ on $[a,1]$ for all $a \in (0,1)$.

2. Let $\mathcal{F} \subset \mathcal{R}[0,1]$ be the set of all Riemann integrable functions on $[0,1]$ such that $|f(t)| \leq M$ for some fixed M. For any $f \in \mathcal{F}$, define $I[f] : [0,1] \to \mathbb{R}$ by

$$I[f](x) = \int_0^{\sqrt{x}} f(t)\, dt.$$

   (a) Show that the family $\{I[f] \mid f \in \mathcal{F}\}$ is equicontinuous.

   (b) Show that given any sequence of functions $\{f_n\}$ in $\mathcal{F}$, there exists a sub-sequence $\{f_{n_k}\}$ such that $I[f_{n_k}]$ converges uniformly on $[0,1]$.

3. Consider the sequence of functions $f_n : [0,2] \to \mathbb{R}$,

$$f_n(t) = \frac{t^n}{1+t^n},$$

   and let $F_n : [0,2] \to \mathbb{R}$ be the anti-derivatives.

   (a) Show that $f_n(t)$ converges point-wise on $[0,2]$. What is the limit function?

   (b) Argue, by simply looking at the limit function above, that no subsequence converges uniformly on $[0,2]$.

   (c) Show that for all $x, y \in [0,2]$,

$$|F_n(x) - F_n(y)| \leq |x-y|.$$

   (d) Show that there is a <u>subsequence</u> F_n that converges uniformly on $[0.2]$.

4. Let $C^0[0,1]$ denote the set of all continuous real valued functions on $[0,1]$. For $f, g \in C^0[0,1]$, define

$$d(f,g) = \sup_{t \in [0,1]} |f(t) - g(t)|.$$

   (a) Show that d defines a metric on $C^0[0,1]$.

   (b) Show that $f_n \to f$ in this metric, if and only if $f_n \to f$ uniformly on $[0,1]$.

(c) Show that $(C^0[0,1], d)$ is a complete metric space, that is every Cauchy sequence is convergent. **Note.** A sequence $\{x_n\}$ in a metric space (X, d) is said to be Cauchy $\forall \varepsilon > 0$, there exists N such that for all $n, m > N$, $d(x_n, x_m) < \varepsilon$. We will talk about completeness in more detail in class on Monday, but this is enough to solve the problem.

5. Let (X, d) be a metric space. The boundary ∂E and frontier dE of a set $E \subset X$ are defined respectively as

$$\partial E = \overline{E} \setminus \text{int}(E),$$
$$dE = \overline{E} \setminus E.$$

where $\overline{E}$ is the closure of the set E and $\text{int}(E)$ is the interior. Consider the following subset of $\mathbb{R}^2$,

$$E = \{(x, y) \mid 0 < x^2 + y^2 < 1\} \cup \{(x, 0) \mid 1 \leq x \leq 2\}.$$

(a) Draw a **neat and labelled** diagram in the x-y plane indicating the subset E. Open sets can be shown with dotted lines.

(b) Write down the sets $\overline{E}$, $\text{int}(E)$, ∂E and dE.

6. If A and B denote arbitrary subsets of a metric space (X, d), prove the following properties.

(a) $\text{int}(A) = X - \overline{X - A}$.

(b) If $\text{int}(A) = \text{int}(B) = \phi$, and A is closed, then $\text{int}(A \cup B) = \phi$. If A is not necessarily closed, given an example where $\text{int}(A \cup B) = X$.

(c) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$. Given an example of strict inclusion.

Here $\text{int}(A)$ denotes the interior of A and $\overline{A}$ as usual denotes the closure.

7. Given $A \subset (X, d)$, let $L(A)$ be the set of limit points of A.

(a) Show that $L(A)$ is closed.

(b) Show that if p is a limit point of $A \cup L(A)$, then p is also a limit point of A. Is it necessarily a limit point of $L(A)$?

8. Let (X, d_X) and (Y, d_Y) be metric spaces. Suppose $f : X \to Y$ is a continuous

(a) For any $y \in Y$, show that $Z_y = \{x \in X \mid f(x) = y\}$ is a closed set.

(b) Suppose now $Y = \mathbb{R}$ with the standard Euclidean metric $|\cdot|$. If for some $p \in X$, $f(p) > 0$, then show that there is some $\delta > 0$ such that for all $x \in B_\delta(p)$, $f(x) > 0$.

9. Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$ be a continuous function. Prove that

$$f(\overline{E}) \subset \overline{f(E)}$$

for any subset $E \subset X$. Show by example that the inclusion can be strict.

10. Show, using only the definition of compactness, that the set

$$K = \{\frac{1}{n} \mid n \in \mathbb{N}\}$$

is NOT compact, while the set $K \cup \{0\}$ is compact.

11. Let $\mathbb{Q}$ be the set of rationals with the usual distance function $d(r, s) = |r - s|$. Let E be the set of rationals r satisfying $2 < r^2 < 3$. Show that E is closed and bounded but not compact. This shows that the Hein-Borel or Bolzano-Weierstrass theorem is not true in a general metric space.

12. Recall that $C^0[0,1]$ denotes the set of continuous functions on $[0,1]$. We endow it with the usual metric

$$d(f,g) = \sup_{t\in[0,1]} |f(t) - g(t)|.$$

Define a function $T : C^0[0,1] \to C^0[0,1]$ by

$$T[f](x) = \int_0^x f(t)\,dt.$$

Let $\mathcal{K} \subset C^0[0,1]$ be a bounded set.

(a) Show that T is a continuous function. Is it injective? **Hint.** To show continuity, it is enough to show (Why?) that if $f_n \xrightarrow{u.c} f$, then $T[f_n] \xrightarrow{u.c} T[f]$.

(b) Show that the set $\overline{T(\mathcal{K})}$ is a compact subset of $C^0[0,1]$. **Hint.** Use the Version-2 of Ascoli-Arzela.

13. This exercise shows that even in a complete metric, a closed and bounded set need not be compact. Let

$$l^\infty(\mathbb{R}); = \{\{a_k\}_{k=1}^\infty \mid a_k \in \mathbb{R}, \text{ and } \sup_k a_k < \infty\}.$$

That is, $l^\infty(\mathbb{R})$ is the set of all *bounded* sequences of real numbers. Note that the M will vary from sequence to sequence. For two sequences $A = \{a_n\}$ and $B = \{b_n\}$, define

$$d(A,B) = \sup_k |a_k - b_k|.$$

(a) For any two sequences $A, B \in l^\infty(\mathbb{R})$, show that $d(A,B)$ is a finite number.

(b) Show that d is a metric on $l^\infty(\mathbb{R})$.

(c) Let E_n be the sequence with 1 at the n^{th} place and zero everywhere else, and let O be the sequence with zeroes everywhere. What is $d(E_n, O)$? $d(E_n, E_m)$ for $n \neq m$?

(d) Show that the set $\overline{B_1(O)}$ is closed and bounded, but not compact. **Hint.** Show that the sequence E_n from above has no limit point.

An application of Arzela-Ascoli to differential equations

The problems in this section are only for the purpose of entertainment, and will not have any bearing whatsoever on your performance in this course.

Our aim (following Rudin, exercise 7.25) is to show that there exists a function $u : [0,1] \to \mathbb{R}$, continuous on $[0,1]$ and differentiable on $(0,1)$ solving the following initial value problem (IVP)

$$\begin{cases} u'(t) = \sin(u(t)), \\ u(0) = c. \end{cases}$$

For a fixed n, and $i = 0, 1, \cdots, n$, put $t_i = i/n$, and let $u_n : [0,1] \to \mathbb{R}$ be the continuous function defined by $u_n(0) = c$ and such that

$$u_n'(t) = \sin(u_n(t_i)), \; t_i < t < t_{i+1}.$$

You should think of u_n as the n^{th} approximation solution to the equation. Essentially, starting at x_0, between x_i and x_{i+1}, the graph of u_n consists of straight line segments with slopes given by $\sin(u_n(x_i))$ (graph the first few functions, say u_1 and u_2). Note that u_n is differentiable everywhere except at $t = t_i$.

Next, define

$$\Delta_n(t) = \begin{cases} u_n'(t) - \sin(u_n(t)), \; t \neq t_i \\ 0, \; \text{otherwise.} \end{cases}$$

So Δ_n measures how far our approximate solutions are from being actual solutions. Moreover, by the definition of Δ_n,

$$u_n(t) = c + \int_0^t [\sin(u_n(t)) + \Delta_n(t)]\, dt.$$

1. Show that on $[0,1]$, $|u_n'(t)| \leq 1$ (wherever it exists), $|\Delta_n(t)| \leq 2$, $\Delta_n(t) \in \mathcal{R}[0,1]$, and $|u_n(t)| \leq |c| + 1$.

2. $\{u_n\}$ is equicontinuous on $[0,1]$. **Note.** You cannot directly apply mean value theorem, since u_n is not differentiable everywhere on $[0,1]$.

3. From this deduce that there exists a subsequence, say $\{u_{n_k}\}$ which converges uniformly to some u on $[0,1]$.

4. Prove that $\sin(u_{n_k}(t)) \xrightarrow{u.c} \sin(u(t))$ on $[0,1]$.

5. From this deduce that $\Delta_{n_k}(t) \xrightarrow{u.c} 0$ on $[0,1]$, since

$$\Delta_n(t) = \sin(u_n(t_i)) - \sin(u_n(t))$$

   on (t_i, t_{i+1}). **Note.** You have to show that the entire sequence $\Delta_n(t)$ converges uniformly to zero, not just $\Delta_{n_k}(t)$.

6. Hence, show that

$$u(t) = c + \int_0^t \sin(u(t))\, dt.$$

   From this, conclude that $u(t)$ solves the initial value problem. Why will this argument not work, if you can only establish pointwise convergence of $\{u_{n_k}\}$?