Assignment-6

(Due 07/30)

1. Let sequences f_n and g_n converge uniformly on some set $E \subset \mathbb{R}$ to f and g respectively
   (a) Construct an example such that $f_n g_n$ does not converge uniformly on E.
   (b) Prove that $f_n g_n$ does converge uniformly if f and g are bounded on E.

2. Show that the sequence of functions
$$f_n(x) = \frac{nx}{n+1}$$
does not converge uniformly on all of $\mathbb{R}$, but does converge uniformly on bounded intervals (a, b). What is the point-wise limit on $\mathbb{R}$?

3. If $g : [0, 1] \to \mathbb{R}$ is a continuous function such that $g(1) = 0$, show that the sequence of functions $\{g(x)x^n\}_{n=1}^{\infty}$ converges uniformly on $[0, 1]$.

4. (a) Define a sequence of functions by
$$f(x) = \begin{cases} 1, & x = 1, \frac{1}{2}, \cdots, \frac{1}{n} \\ 0, & \text{otherwise.} \end{cases}$$
Calculate the pointwise limit function f. Is each f_n continuous at zero? Does $f_n \to f$ uniformly on $\mathbb{R}$. Is f continuous at 0?
   (b) Repeat the exercise with the functions
$$g(x) = \begin{cases} x, & x = 1, \frac{1}{2}, \cdots, \frac{1}{n} \\ 0, & \text{otherwise,} \end{cases}$$
and
$$h(x) = \begin{cases} x, & x = 1, \frac{1}{2}, \cdots, \frac{1}{n-1} \\ 1, & x = \frac{1}{n} \\ 0, & \text{otherwise.} \end{cases}$$

5. Let $\{f_n\}$ be a sequence of real-valued continuous functions $f_n : E \to \mathbb{R}$ for some subset $E \subset \mathbb{R}$. Suppose $f_n \xrightarrow{u.c} f$ on E. Show that
$$f_n(x_n) \to f(x)$$
for every sequence of points $x_n \to x$ in E. Is the conclusion true if $f_n \to f$ only pointwise? Either provide a proof, or a counter example. Is the converse of the above statement true?

6. For $n = 1, 2, \cdots$ and $x \in \mathbb{R}$, define
$$f_n(x) = \frac{x}{1 + nx^2}.$$
Show that $\{f_n\}$ converges uniformly to a differentiable function f on $\mathbb{R}$, and that the equation
$$f'(x) = \lim_{n\to\infty} f_n'(x)$$
is correct for all $x \neq 0$ but false at $x = 0$. Why does this not contradict the theorem on uniform convergence and differentiation?

7. Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of bounded functions. That is for each n, there exists an M_n such that

$$|f_n(t)| < M_n$$

for all $t \in [0,1]$. Suppose in addition that $f_n \xrightarrow{u.c} f$.

(a) Show that f is bounded on $[0,1]$. **Hint.** Apply the definition of uniform convergence with $\varepsilon = 1$.

(b) Show that the sequence of functions is *uniformly bounded*. That is, show that there is an M such that

$$|f_n(t)| < M$$

for all $t \in [0,1]$ and all n. **Hint.** Show that for large n, f_n can be bounded essentially by the bound for f.

(c) Suppose additionally $f_n \in \mathcal{R}[0,1]$ for all n. Prove or disprove -

$$\lim_{n\to\infty} \int_0^{1-1/n} f_n(t)\,dt = \int_0^1 f(t)\,dt.$$

Hint.

$$\int_0^{1-1/n} f_n(t)\,dt = \int_0^1 f_n(t)\,dt - \int_0^{1/n} f_n(t)\,dt$$

8. Find the radius and interval of convergence of the following series.

1. $\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n}$
2. $\sum_{n=1}^{\infty} \frac{2^n}{n^2} x^n$.
3. $\sum_{n=0}^{\infty} \frac{2^n}{n!} x^n$.

9. Decide whether each proposition is true or false, providing a complete proof, or a counter example, as appropriate.

(a) If $\sum f_n$ converges uniformly, then f_n converges uniformly to zero.

(b) If $0 \le f_n(x) \le g_n(x)$ and $\sum g_n$ converges uniformly, then $\sum f_n$ also converges uniformly.

(c) If $\sum f_n$ converges uniformly on E, then there exists constants M_n such that $|f_n(x)| < M_n$ for all $x \in E$ and $\sum M_n$ converges.

(d) If each f_n is uniformly continuous on E and $f_n \xrightarrow{u.c} f$ on E, then f is also uniformly continuous on E.

(e) If f_n has a finite number of discontinuities on E and $f_n \xrightarrow{u.c} f$, then f has a finite number of discontinuities on E.

(f) If f_n has at most M number of discontinuities on E (where M is fixed and independent of n) and $f_n \xrightarrow{u.c} f$, then f has at most M number of discontinuities on E.

10. Define

$$g(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{x^{2n}+1}.$$

Find the values of x where the series converges, and show that we get a continuous function on this set.

11. Let

$$h(x) = \sum_{n=1}^{\infty} \frac{1}{x^2+n^2}.$$

(a) Show that h is continuous on all of $\mathbb{R}$.

(b) Is h differentiable? If so, is the derivative h' continuous? Give complete proofs.

12. We saw in class that a function represented by a power series is automatically smooth, that is, it has derivatives of all orders. The aim of this question to show that the converse might not be true. That is, there exist smooth functions that cannot be represented by a power series.

    (a) If $P(x)$ is a polynomial, show that

    $$\lim_{x\to\infty} P(x)e^{-x^2} = 0.$$

    Hint. Use L'Hospital's rule and an induction on the degree of the polynomial.

    (b) Now define

    $$f(t) = \begin{cases} e^{-1/t^2}, & t \neq 0 \\ 0, & t = 0. \end{cases}$$

    Show that f has derivatives of all orders at $x = 0$, and that $f^{(n)}(0) = 0$ for all $n = 1, 2, \cdots$. Can f be represented by a power series in a neighborhood of $t = 0$?

Periodicity of sine and cosine.

This exercise is unimportant from the point of view of doing well in the course, but highly recommended for those with a wish to explore the non-trivial origins of one of the most trivial of high-school facts.

Recall that $\sin x, \cos x : \mathbb{R} \to \mathbb{R}$ are defined by the power series

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$
$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.$$

In class, we saw that as a consequence, we have the following basic properties:

$$\begin{aligned} \frac{d \sin x}{dx} = \cos x, \ \frac{d \cos x}{dx} &= \sin x \\ \sin(-x) = \sin(x), \ \cos(-x) &= \cos(x) \\ \sin(x+y) &= \sin x \cos y + \cos x \sin y \\ \cos(x+y) &= \cos x \cos y - \sin x \sin y \\ \sin^2 x + \cos^2 x &= 1. \end{aligned}$$

Our aim is to prove the following theorem.

Theorem. There exists a real number $\pi > 0$ such that $\sin(x + 2\pi) = \sin(x)$ for all $x \in \mathbb{R}$. Moreover, if $\beta \in \mathbb{R}$ such that $\sin(x + \beta) = \sin(x)$ for all x, then $\beta = 2n\pi$ for some $n \in \mathbb{Z}$.

Proof.

(i) Show that if $0 \leq x \leq \sqrt{2}$, then for all $n = 0, 1, 2, \cdots$,

$$\frac{x^{4n}}{(4n)!} - \frac{x^{4n+2}}{(4n+2)!} \geq 0.$$

Hence show that $\cos x > 0$ if $x \in [0, \sqrt{2}]$. In particular, this shows that there is no root of $\cos x$ in $[0, \sqrt{2}]$.

(ii) Show that $\cos 2 < -\frac{1}{3}$, and hence show that there is at least one root of $\cos x$ in $[\sqrt{2}, 2]$.

(iii) Next, show that $\sin x \geq \frac{x}{3}$ when $x \in [0, 2]$. Use this to show that $|\sin x| \geq \frac{|x|}{3}$ for all $x \in [-2, 2]$.

(iv) Use this and the addition formulas to conclude that $\cos x$ has a unique root in $[\sqrt{2}, 2]$. **Hint.** If $\sqrt{2} \leq x_1 < x_2 \leq 2$ were two roots, then show that $\sin(x_2 - x_1)$ would have to be zero. But this should contradict the above inequality.

(v) Let this unique root be ζ and define $\pi = 2\zeta$. Show that $\cos n\pi = (-1)^n$ for all integers n. In particular, show that $\cos 2n\pi = 1$.

(vi) Hence, show that $\sin(x + 2\pi) = \sin(x)$ and $\cos(x + 2\pi) = \cos(x)$. It now remains to show that any other period has to be an integer multiple of 2π.

(vii) Show that if $\sin(x + \beta) = \sin x$ for all x, then $\sin(\beta/2) = 0$ and hence $\cos \beta = 1$.

(viii) To finish off the proof of the theorem, show that if $\cos \beta = 1$, then $\beta = 2n\pi$ for some integer n. **Hint.** Without loss of generality, let $\beta > 0$. There is a natural number n such that $-\pi \leq \beta - 2n\pi < \pi$. Show that $\sin(\beta/2 - n\pi) = 0$ but that this contradicts the inequality in (iii) unless $\beta = 2n\pi$.