Assignment-5

(Due 07/23)

Only submit the questions in red.

1. Suppose f is a bounded real valued function on $[a,b]$ such that $f^2 \in \mathcal{R}[a,b]$. Does it follow that $f \in \mathcal{R}[a,b]$? Does the answer change if we assume $f^3 \in \mathcal{R}[a,b]$? Either give a proof or provide a counter example in each of the two cases.

2. Let
$$f(x) = \begin{cases} x^2, & x \in \mathbb{Q} \\ 0, & \text{otherwise.} \end{cases}$$
   (a) Calculate the upper and lower integrals $U(f)$ and $L(f)$ for f on $[0,b]$.
   (b) Is f integrable on $[0,b]$. Answer the question, solely based on your calculations in part(a), and not by quoting a theorem that we might have learnt in class.

3. Let $f : [0,1] \to \mathbb{R}$ be defined by
$$f(t) = \begin{cases} 2^{-n}, & 2^{-n-1} < t \leq 2^{-n} \\ 0, & t = 0. \end{cases} \quad .$$
   Show that $f \in \mathcal{R}[0,1]$ by showing that given any $\varepsilon > 0$, there exists a partition P such that
$$U(P,f) - L(P,f) < \varepsilon,$$
   and without appealing to the theorem of Lebesgue.

4. (a) Let $f \in \mathcal{R}[a,b]$ and $\{p_1, \cdots p_n\}$ be a finite collection of points in $[a,b]$. Let $g : [a.b] \to \mathbb{R}$ be a bounded function such that
$$f(t) = g(t),$$
   for all $t \in [a,b] \setminus \{p_1, \cdots, p_n\}$. Show that $g \in \mathcal{R}[a,b]$ and that
$$\int_a^b g(t)\,dt = \int_a^b f(t)\,dt.$$
   Hint. Do it for one point at a time.
   (b) Is the conclusion true, if we instead have a countable collection of points $\{p_n\}_{n=1}^{\infty}$? **Hint.** What is the most basic non-integrable function that you know?

5. (a) If $f \in \mathcal{R}[0,1]$, show that
$$\int_0^1 f(t)\,dt = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\Big(\frac{k}{n}\Big).$$
   (b) Give an example of a bounded function $f : [0,1] \to \mathbb{R}$ for which the limit on the right exists, but f is not Riemann integrable.

(c) Use part(a) to evaluate the limit

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \frac{1}{\sqrt{k}}.$$

6. (a) Let f be a continuous real valued function on $[a, b]$. Show that there exists a $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(t)\, dt.$$

(b) More generally, if f is continuous on $[a, b]$, $g \in \mathcal{R}[a, b]$ and g does not change sign (you can assume $g \geq 0$), then prove that exists a $c \in [a, b]$ such that

$$\int_a^b f(t)g(t)\, dt = f(c) \int_a^b g(t)\, dt.$$

Hint. Let $I = \int_a^b g(t)\, dt \neq 0$ and $f[a, b] = [m, M]$. The proof is easy if $I = 0$. If $I \neq 0$, show that

$$m < \frac{1}{I} \int_a^b f(t)g(t)\, dt < M,$$

and use intermediate value theorem.

7. Let $f : [1, \infty) \to (0, \infty)$ be a continuous, decreasing function such that $\lim_{x\to\infty} f(x) = 0$. Denote

$$s_n = \sum_{k=1}^{n} f(k),\ I_n = \int_1^n f(t)\, dt,\ d_n = s_n - I_n.$$

(a) Show that $f(n) + I_n \leq s_n \leq f(1) + I_n$.

(b) (Integral test for convergence) Hence show that $\sum_{n=1}^{\infty} f(n)$ converges if and only if $\int_1^{\infty} f(x)\, dx$ converges.

(c) Use the above test, to find all possible values of p and q for which the following series converge.

1. $\sum_{n=2}^{\infty} \frac{1}{n^p (\ln n)^q}$

2. $\sum_{n=3}^{\infty} \frac{1}{n \ln n (\ln \ln n)^p}$.

8. (a) Let $f : [a, b] \to \mathbb{R}$ be continuous function, and let $p, q : [c, d] \to [a, b]$ be continuous functions, differentiable on the interior (c, d). Define

$$F(x) = \int_{p(x)}^{q(x)} f(t)\, dt.$$

Show that F is continuous on $[c, d]$ and differentiable on (c, d), and that

$$F'(x) = f(q(x))q'(x) - f(p(x))p'(x).$$

Hint. Write F as a composition of two functions, one of which can be differentiated using the fundamental theorem of calculus. Then properties of F follow from corresponding properties of compositions.

(b) Now, let $F : (0, \pi) \to \mathbb{R}$ be defined by

$$F(x) = \int_{\sin x}^{1} \ln t\, dt.$$

Calculate $F'(\pi/4)$ and $F'(\pi/2)$ in two ways. First, by evaluating the integral, and then differentiating. And second, by using part(a) above. Your answers should of course be the same.

9. (a) Let $f(x) = |x|$, and define $F(x) = \int_{-1}^{x} f(t)\,dt$. Find a piecewise algebraic formula for $F(x)$. Where is F continuous? Where is it differentiable? Where does $F' = f$?

   (b) Now repeat part(a) with

$$f(x) = \begin{cases} 1, & x < 0 \\ 2, & x \geq 0. \end{cases}$$

10. Calculate $\lim_{x\to 0} \frac{1}{x} \int_0^x e^{t^2}\,dt$. Give complete justifications, quoting any theorems that might have been used.

11. Find the set of all values of p for which the following improper integrals converge.

    1. $\int_0^1 \frac{1-\sin x}{x^p}$.
    2. $\int_0^\infty \frac{\ln(1+x)}{x^p}$.

    Hint. Taylor's theorem might be useful while analyzing the integrands near $x = 0$.

12. (a) Show that for $n = 1, 2, 3, \cdots$

$$\int_{(n-1)\pi}^{n\pi} \left|\frac{\sin x}{x}\right| dx \geq \frac{2}{n\pi}.$$

   (b) Hence show that $\int_\pi^\infty \left|\frac{\sin x}{x}\right| dx$ diverges.

   (c) For any $R > 0$, show that

$$\int_\pi^R \frac{\sin x}{x}\,dx = \frac{1}{\pi} - \frac{\cos R}{R} - \int_\pi^R \frac{\cos x}{x^2}\,dx.$$

   (d) Hence, show that $\int_\pi^\infty \frac{\sin x}{x}\,dx$ is a convergent integral.

Remark In fact, one can show that

$$\int_0^\infty \frac{\sin x}{x} = \frac{\pi}{2}.$$

This formula is usually one of the high points of a course in complex analysis, and is a consequence of the so-called residue theorem (yes, a real integral evaluated using complex numbers!). But there are are several, real-variable proofs of this, including using doubles integrals or using differentiation under the integral sign (popularized by Feynman, as an alternative to residue calculus).