Assignment-4

(not to be handed in)

1. Show that if f is differentiable at $x = p$, then
$$\lim_{h\to 0} \frac{f(p+h) - f(p-h)}{2h} = f'(p).$$

2. Let f and g be differentiable functions on (a, b) and let $p \in (a, b)$. Define
$$h(t) = \begin{cases} f(t), & t \in (a, p) \\ g(t), & t \in [p, b). \end{cases}$$
Show that h is differentiable on (a, b) if and only if $f(p) = g(p)$ and $f'(p) = g'(p)$.

3. (a) Show that $|\sin\theta| \leq |\theta|$, for all $\theta \in \mathbb{R}$.
   (b) More generally, show that if $g : \mathbb{R} \to \mathbb{R}$ is differentiable such that $|g'(t)| \leq M$ and $g(0) = 0$, then
   $$|g(t)| \leq M|t|,$$
   for all $t \in \mathbb{R}$.

4. (a) Show that $\tan x > x$ for all $x \in (0, \pi/2)$.
   (b) Show that
   $$\frac{2x}{\pi} < \sin x < x$$
   for all $x \in [0, \pi/2]$. **Hint.** Consider the function $\sin x/x$. Is it monotonic?

5. Find the following limits if they exist.
   1. $\lim_{x\to 0} \frac{x - \sin x}{x^3}$
   2. $\lim_{x\to 0} \frac{1 - \cos 2x - 2x^2}{x^4}$
   3. $\lim_{x\to\infty} (e^x + x)^{1/x}$
   4. $\lim_{x\to 0} (\cos x)^{1/x^2}$
   5. $\lim_{x\to 0^+} \frac{1 - \cos x}{e^x - 1}$
   6. $\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$

6. Consider the functions
$$f(x) = x + \cos x \sin x \text{ and } g(x) = e^{\sin x}(x + \cos x \sin x).$$
   (a) Show that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$.
   (b) Show that if $\cos x \neq 0$ and $x > 3$, then
   $$\frac{f'(x)}{g'(x)} = \frac{2e^{-\sin x} \cos x}{2\cos x + f(x)}.$$
   (c) Show that
   $$\lim_{x\to\infty} \frac{2e^{-\sin x} \cos x}{2\cos x + f(x)} = 0,$$
   and yet, the limit $\lim_{x\to} \frac{f(x)}{g(x)}$ does not exist.

(d) Explain why this does not contradict L'Hospital's rule.

7. (a) Show that $e^x \geq 1 + x$ for all $x \in \mathbb{R}$.
   (b) Show that there exists a constant $M > 0$ such that
$$|\frac{e^x - 1 - x}{x^2} - \frac{1}{2}| \leq M|x|,$$
   for all $x \in [-1, 1] \setminus \{0\}$. **Hint.** Taylor's thoerem.
   (c) Compute
$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}.$$

8. Show the following *Bernoulli's* inequalities.
   (a) If $r \in [0, 1]$ and $x \geq -1$, show that
$$(1 + x)^r \leq 1 + rx.$$
   (b) If $r \in (-\infty, 0) \cup (1, \infty)$, and $x \geq -1$, show that
$$(1 + x)^r \geq 1 + rx.$$
   Hint. You can either use the try to find the local max or min, or simply use the fact that if $f' \geq 0$, then f is increasing.

9. Suppose $f \in C^5[-1, 1]$, such that $f(0) = 1$, and $f'(0) = \cdots = f^4(0) = 0$. If $f^5(0) < 0$, show that there exists a $\delta > 0$ such that
$$f(x) < 1,$$
   for all $x \in (0, \delta)$.

10. A function $f : E \to \mathbb{R}$ is called *Lipschitz* (or more precisely M-Lipschitz) if there exists an $M > 0$ such that for all $x, y \in E$,
$$|f(x) - f(y)| \leq M|x - y|.$$
   (a) Show that any Lipschitz function is uniformly continuous.
   (b) Show that if $f : (a, b) \to \mathbb{R}$ is a differentiable function such that $|f'(t)| \leq M$ for all $t \in (a, b)$, then f is M-Lipschitz.
   (c) Let $f : \mathbb{R} \to \mathbb{R}$ be a *contraction*, that is an α-Lipschitz function, for some $\alpha < 1$. Show that there exists a *fixed point* p, that is, a $p \in \mathbb{R}$ such that $f(x) = x$. **Hint.** Let $x_0 \in \mathbb{R}$ be any real number. Having chosen $x_0, x_1, \cdots, x_n$, let $x_{n+1} = f(x_n)$. Show that $\{x_n\}$ is a Cauchy sequence, and hence must converge, and that the limit p must satisfy $f(p) = p$.
   (d) Show that the fixed point so obtained will be unique.

11. For $\alpha > 0$, a function $f : E \to \mathbb{R}$ is said to be *α-Hölder*, if
$$|f(x) - f(y)| \leq M|x - y|^\alpha,$$
   for all $x, y \in E$ and some $M > 0$.
   (a) Show that any α-Hölder function is uniformly continuous.
   (b) Show that if $f : (a, b) \to \mathbb{R}$ is α-Hölder for some $\alpha > 1$, then f is differentiable, and is in fact a constant function.

12. Assume that f has a finite derivative on (a, ∞).
   (a) If $f(x) \to 1$ and $f'(x) \to c$ as $x \to \infty$, prove that $c = 0$. **Hint.** Show, using the mean value theorem, that there is a sequence $x_n \in (n, n + 1)$ such that $f'(x_n) \to 0$.
   (b) If $f'(x) \to 1$ as $x \to \infty$, prove that
$$\lim_{x \to \infty} \frac{f(x)}{x} = 1.$$