Assignment-3

(Due 07/09)

Only submit the questions in red.

1. (a) Let $f : (a, b) \to \mathbb{R}$ be continuous such that for some $p \in (a, b)$, $f(p) > 0$. Show that there exists a $\delta > 0$ such that $f(x) > 0$ for all $x \in (p - \delta, p + \delta)$.

   (b) Let $E \subset \mathbb{R}$ be a subset such that there exists a sequence $\{x_n\}$ in E with the property that $x_n \to x_0 \notin E$. Show that there is an unbounded continuous function $f : E \to \mathbb{R}$.

2. (a) If $a, b \in \mathbb{R}$, show that

$$\max\{a, b\} = \frac{(a + b) + |a - b|}{2}.$$

   (b) Show that if $f_1, f_2, \cdots, f_n$ are continuous functions on a domain $E \subset \mathbb{R}$, then

$$g(x) = \max\{f_1(x), \cdots, f_n(x)\}$$

   is again a continuous function on E.

   (c) Let's explore if the infinite version of this true or not. For each $n \in \mathbb{N}$, define

$$f_n(x) = \begin{cases} 1, & |x| \geq 1/n \\ n|x|, & |x| < 1/n. \end{cases}$$

   Explicitly compute $h(x) = \sup\{f_1(x), f_2(x), \cdots, f_n(x), \cdots\}$. Is it continuous?

3. For each of the following, decide if the function is uniformly continuous or not. In either case, give a proof using just the definition in terms of ε and δ.

   (a) $f(x) = \sqrt{x^2 + 1}$ on $(0, 1)$.

   (b) $g(x) = x \sin(1/x)$ on $(0, 1)$.

   (c) $g(x) = \frac{1}{x^2}$ on $[1, \infty)$.

   (d) $g(x) = \frac{1}{x^2}$ on $(0, 1]$

4. (a) Let $f : E \to \mathbb{R}$ be uniformly continuous. If $\{x_n\}$ is a Cauchy sequence in E, show that $\{f(x_n)\}$ is also a Cauchy sequence.

   (b) Show, by exhibiting an example, that the above statement is not true if f is merely assumed to be continuous.

   (c) Let $f : (a, b) \to \mathbb{R}$ be continuous. Show that there exists a continuous function $F : [a, b] \to \mathbb{R}$ such that $F(x) = f(x)$ for all $x \in (a, b)$ if and only if f is uniformly continuous. **Hint.** Given f, how should you define $F(a)$ and $F(b)$?

5. (a) Show directly from the definition of uniform continuity, that any uniformly continuous function $f : (a, b) \to \mathbb{R}$ is bounded.

   (b) If $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous, show that there exist $A, B \in \mathbb{R}$ such that $|f(x)| \leq A|x| + B$ for all $x \in \mathbb{R}$. **Hint.** Again apply the definition of uniform continuity with $\varepsilon = 1$. For the corresponding $\delta > 0$, note that any $x \in \mathbb{R}$ can be reached from 0 be a sequence of roughly $|x|/\delta$ steps. Now apply the triangle inequality repeatedly to compare $|f(x)|$ with $|f(0)|$.

6. Let $f : [0, 1] \to \mathbb{R}$ be continuous with $f(0) = f(1)$.
   (a) Show that there must exist $x, y \in [0, 1]$ satisfying $|x - y| = 1/2$ such that $f(x) = f(y)$. **Hint.** Consider the function $g(x) = f(x + 1/2) - f(x)$ on $[0, 1/2]$ and use intermediate value theorem.
   (b) Show that for each $n \in \mathbb{N}$, there exist $x_n, y_n \in [0, 1]$ such that $f(x_n) = f(y_n)$. **Hint.** Again consider $g_n(x) = f(x + 1/n) - f(x)$ on $[0, \frac{n-1}{n}]$. Can $g_n(k/n)$ all have the same sign for $k = 0, 1, \cdots, \frac{n-1}{n}$?
   (c) On the other hand, if $h \in [0, 1/2]$ is not of the form $1/n$, show that there does not necessarily exist x, y such that $|x - y| = h$ with $f(x) = f(y)$. Give an example with $h = 2/5$.

7. For each stated limit, and ε, find the largest possible δ-neighborhood that makes the definition of limits work.
   (a) $\lim_{x \to 4} \sqrt{x} = 2, \ \varepsilon = 1.$
   (b) $\lim_{x \to \pi} \lfloor x \rfloor = 3, \ \varepsilon = 0.01.$

8. Compute each limit or state that it does not exist. Use any of the tools to justify your answer.
   (a) $\lim_{x \to 2} \frac{|x - 2|}{x - 2}.$
   (b) $\lim_{x \to 0} \sqrt[3]{x}(-1)^{\lfloor 1/x \rfloor}$

9. Recall that every rational number x can be written as m/n, where $n > 0$ and $gcd(m, n) = 1$. When $x = 0$, we take $m = 0$ and $n = 1$. Consider $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0, & x \text{ is irrational} \\ \frac{1}{n}, & x = \frac{m}{n}. \end{cases}$$

   (a) Show that for any real number α and and integer N, there exists a $\delta > 0$ such that every rational number in the interval $(\alpha - \delta, \alpha + \delta)$, not equal to α, has denominator greater than N. **Hint.** First show that the number of rational numbers in $(\alpha - 1, \alpha + 1)$ with denominator smaller than N is finite. Then choose $\delta < 1$ small enough to exclude all these rationals.
   (b) For any real number α, show that $\lim_{t \to \alpha} f(t) = 0$.
   (c) Prove that f is continuous at every irrational number, and has a removable discontinuity at every rational number.

10. Suppose a and c are real numbers, $c > 0$, and $f : [-1, 1] \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} x^a \sin(|x|^{-c}) \ (x \neq 0), \\ 0 \ (x = 0). \end{cases}$$

   Prove the following statements.
   (a) f is continuous if and only if $a > 0$.
   (b) $f'(0)$ exists if and only if $a > 1$.
   (c) $f'(x)$ is bounded if and only if $a \geq 1 + c$.
   (d) $f'(x)$ is continuous if and only if $a > 1 + c$.