Assignment-2

(Due 07/02)

Only submit the questions in red.

1. (a) For any two sequences $\{a_n\}$ and $\{b_n\}$ show that

$$\limsup_{n\to\infty}(a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n,$$

unless the right hand side is of the form $\infty - \infty$.

(b) Find sequences $\{a_n\}$ and $\{b_n\}$ with strict inequality above.

2. Let $\{a_n\}$ be a sequence of real numbers, and let

$$S = \{x \in \mathbb{R} \mid \exists \text{ a sub-sequence } a_{n_k} \text{ such that } a_{n_k} \xrightarrow{k\to\infty} x\}.$$

(a) Show that $L = \limsup a_n$ if and only if $L = \sup S$.

(b) Formulate and prove the analogous statement for $\liminf$.

Note. From now on, you can use the conclusions of this exercise as a theorem. So now, you have a definition of $\limsup$ and two other equivalent characterizations.

3. Find the $\limsup$ and $\liminf$ of the sequence $\{a_n\}$ defined recursively by

$$a_1 = 0,\ a_{2m} = \frac{a_{2m-1}}{2},\ a_{2m+1} = \frac{1}{2} + a_{2m}.$$

Justify your answers with complete proofs.

4. (a) Let $\{a_n\}$ be a bounded sequence with the property that every convergent subsequence converges to the same limit a. Show that the entire sequence $\{a_n\}$ converges and $\lim_{n\to\infty} a_n = a$.

(b) Now assume that $\{a_n\}$ is a sequence with the property that every subsequence has a further subsequence that converges to the same limit a. Show that the entire sequence $\{a_n\}$ converges and $\lim_{n\to\infty} a_n = a$.

5. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of real numbers satisfying

$$|a_{n+1} - a_n| \le \frac{1}{2}|a_n - a_{n-1}|.$$

Show that the sequence converges. **Hint.** Show that the sequence is Cauchy.

6. Let $S = \{n_1, n_2, \cdots\}$ denote the collection of those positive integers that do not have the digit 0 in their decimal representation. (For example $7 \in S$ but $101 \notin S$). Show that $\sum_{k=1}^{\infty} 1/n_k$ converges. **Note.** This should be a surprising result in that leaving out only a few (but of course still infinite) terms out of the harmonic series, we end up with a series that suddenly converges.

7. The *Fibonacci numbers* $\{f_n\}$ are defined by

$$f_0 = f_1 = 1,\ \text{and } f_{n+1} = f_n + f_{n-1} \text{ for } n = 1, 2, \cdots.$$

For $n = 1, 2, \cdots$, we also define $r_n = f_{n+1}/f_n$.

(a) Find a formula for r_{n+1} in terms of r_n.

(b) Show that $f_n \geq n$ for all $n \geq 2$.

(c) Show that $f_{n+1}f_{n-1} - f_n^2 = (-1)^{n+1}$.

(d) Hence show that if $n \geq 2$, then

$$|r_{n+1} - r_n| \leq \frac{1}{(n-1)^2}.$$

(e) Hence show that the sequence of ratios $\{r_n\}$ converge, and compute it's limit. **Note.** This limit is the so-called *golden ratio.*

8. Investigate the behavior of each series (convergence, divergence, conditional convergence, absolute convergence). In cases that there is a parameter (p, q or r) find the range of values where the series exhibits the above behavior.

1. $\sum_{n=1}^{\infty} p^n n^p$ $(p > 0)$
2. $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n+1}-\sqrt{n}}{n^p}$
3. $\sum_{n=1}^{\infty} \frac{1}{p^n - q^n}$, $(0 < q < p)$
4. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$
5. $\sum_{n=1}^{\infty} (\sqrt[n]{n} - 1)^n$
6. $\sum_{n=1}^{\infty} \frac{1}{1+r^n}$.

9. (a) Let $\{a_n\}$ be a sequence of of positive real numbers. Show that

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \leq \liminf_{n\to\infty} \sqrt[n]{a_n} \leq \limsup_{n\to\infty} \sqrt[n]{a_n} \leq \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

You may assume that each of the quantities is finite, even though the result holds true for extended reals. **Hint.** Proceed by contradiction. For instance, for the rightmost inequality, let $U = \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}$ and $L = \limsup_{n\to\infty} \sqrt[n]{a_n}$ and suppose $L > U$. Then use the equivalent characterizations of $\limsup$ to draw a contradiction.

(b) Show that if $\sum a_n$ converges by the ratio test, then $\sum a_n$ also converges by the root test.

(c) Consider the sequence $\{a_n\}_{n=0}^{\infty}$,

$$a_n = \frac{1}{2^{n+(-1)^n}} = \begin{cases} \frac{1}{2^{n-1}}, & n \text{ is odd} \\ \frac{1}{2^{n+1}}, & n \text{ is even.} \end{cases}$$

Compute (with proper justifications) $\limsup \sqrt[n]{|a_n|}$ and $\limsup |a_{n+1}/a_n|$. Show that the series converges by the root test. Does the ration test work?

(d) Let $b_n = n^n/n!$. Show that

$$\lim_{n\to\infty} \sqrt[n]{b_n} = e.$$

Hint. It is easier to compute the limiting ratios.

10. (a) Show that if $a_n > 0$, and $\lim_{n\to\infty} na_n = l \neq 0$, then $\sum a_n$ diverges.

(b) Given that $\sum a_n$ converges *absolutely*, show that $\sum a_n^p$ also converges whenever $p > 1$. Give a counterexample, if $\sum a_n$ only converges conditionally.

11. Consider each of the following propositions. Provide short proofs for those that are true and counterexamples for any that are not.

(a) If $\sum a_n$ converges and the sequence $\{b_n\}$ also converges, then $\sum a_n b_n$ converges.

(b) If $\sum a_n$ converges conditionally, then $\sum n^2 a_n$ diverges.

(c) If $\{a_n\}$ is a decreasing sequence, and $\sum a_n$ converges, then $\lim_{n\to\infty} na_n = 0$.

12. (a) For any $n \in \mathbb{N}$, show that the function $p_n(x) = x^n$ is continuous on all of $\mathbb{R}$. Show the explicit dependence of δ on ε and the point that you are looking at.

(b) Show that $f(x) = \sqrt{x}$ is continuous on $(0, \infty)$.

(c) Show that $f_n(x) = x^{1/n}$ is continuous on $(0, \infty)$.

Hint. For all parts the following identity might be useful.

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}).$$