Assignment-1

(Due 06/25)

In each of the following, only use (and indicate) theorems or axioms introduced in the lectures. Only hand in the problems in red.

1. (a) Show that $\sqrt{12}$ is irrational.

   (b) Now consider the set $E := \{\alpha \in \mathbb{Q} \mid \alpha^2 < 12\}$. Given any $\beta \in \mathbb{Q}$ such that $\beta^2 < 12$, find an explicit rational number $\varepsilon > 0$ (depending of course on β), such that $(\beta + \varepsilon)^2 < 12$. **Hint.** It is easy to see that $\beta < 4$. Also if ε is chosen less than 1, then $\varepsilon^2 < \varepsilon$ (which order axiom are we using?).

   (c) Similarly, if $\beta^2 > 12$, and $\beta < 4$, find an explicit positive rational number ε such that $(\beta - \varepsilon)^2 > 12$ and yet $\beta - \varepsilon$ is an upper bound of E.

   (d) Hence show that E has no least upper bound in $\mathbb{Q}$.

2. Let $A, B \subset \mathbb{R}$.

   (a) If $\sup A < \sup B$, then show that there is some $b \in B$ which is an upper bound for A.

   (b) Show, by providing an example, that this is not necessarily the case if $\sup A \leq \sup B$.

3. Let $a < b$ be real numbers, and consider the set $T = \mathbb{Q} \cap [a, b]$. Show that $\inf T = a$ and $\sup T = b$.

4. Let $a, b \in \mathbb{R}$.

   (a) Show that $|b| \leq a$ if and only if $-a \leq b \leq a$.

   (b) Show that $||b| - |a|| \leq |b - a|$.

5. (a) Let $a, b \in \mathbb{R}$ such that $a \leq b + \frac{1}{n}$ for all $n \in \mathbb{N}$. Show that $a \leq b$.

   (b) Show that if $a > 0$, then there exists a natural number $n \in \mathbb{N}$ such that $\frac{1}{n} \leq a \leq n$.

   (c) Let $a, b \in \mathbb{R}$ such that $a < b$. Use the denseness of $\mathbb{Q}$ to show that there are infinitely many rationals between a and b.

6. * Let A and B be non-empty subsets of $\mathbb{R}$, and let

   $$A + B := \{a + b \mid a \in A,\ b \in B\}.$$

   That is, $A + B$ is the set of all sums $a + b$, where $a \in A$ and $b \in B$.

   (a) Show that $\sup(A + B) = \sup A + \sup B$. **Note.** You need to separately consider the case when at least one of the two supremums on the right is ∞.

   (b) $\inf(A + B) = \inf A + \inf B$.

7. For each sequence, find the limit, and use the definition of limits to prove that the sequence does indeed converge to the proposed limit. Note that this means, given an $\varepsilon > 0$, you need to write down an N for which the definition of convergence works. Try to make the dependence of N on ε as explicit as possible.

   (a) $\displaystyle\lim_{n\to\infty} \frac{3n+1}{6n+5}$.

   (b) $\displaystyle\lim_{n\to\infty} \left\lfloor \frac{12+4n}{3n} \right\rfloor$, where for any $x \in \mathbb{R}$, the floor function $\lfloor x \rfloor$ is the greatest integer smaller than or equal to x (for instance, $\lfloor \pi \rfloor = 3$, $\lfloor -2.3 \rfloor = -3$).

8. Give an example of each of the following or state that such a request is impossible by referencing the correct theorem or giving a complete proof.
   (a) Sequences $\{x_n\}$ and $\{y_n\}$ that both diverge, but $\{x_n + y_n\}$ converges.
   (b) Sequence $\{x_n\}$ converges and $\{y_n\}$ diverges, but $\{x_n + y_n\}$ converges.
   (c) Two sequences $\{x_n\}$ and $\{y_n\}$ where $\{x_n y_n\}$ and $\{x_n\}$ converge, but $\{y_n\}$ diverges.
   (d) Let $k \in \mathbb{N}$ be fixed. Then sequence $\{a_n\}_{n=1}^{\infty}$ converges but $\{a_{n+k}\}_{n=1}^{\infty}$ might not converge, or even if it converges, might not converge to the same limit.

9. Let $\{x_n\}$ and $\{y_n\}$ be two sequences, and let $\{z_n\}$ by the "shuffled" sequence $\{x_1, y_1, x_2, y_2, \cdots\}$.
   (a) Find a general formula for z_n.
   (b) Show that $\{z_n\}$ converges if and only if **both** $\{x_n\}$ and $\{y_n\}$ converge to the same value.

10. Let $a_1 = 1$ and

$$a_{n+1} = \frac{1}{3}(a_n + 1)$$

for $n > 1$.
   (a) Find a_2, a_3 and a_4.
   (b) Use induction to show that $a_n > 1/2$ for all n.
   (c) Show that $\{a_n\}$ is a convergent sequence and compute it's limit.

11. (a) Show that the sequence

$$\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \cdots$$

   converges, and find it's limit.
   (b) Does the sequence $\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}$ converge? Give a complete proof. If it does converge, also find the limit.

12. (Arithmetic and geometric means)
   (a) Show that

$$\frac{x+y}{2} \geq \sqrt{xy}.$$

   The quantity on the left is the arithmetic mean, and the quantity on the right is the geometric mean.
   (b) Now, let $0 \leq x_1 \leq y_1$, and define recursively,

$$x_{n+1} = \sqrt{x_n y_n}, \;\; y_{n+1} = \frac{x_n + y_n}{2}.$$

   Show that both $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} y_n$ exist and are equal.