Assignment-0

(not to be handed in)

1. (De Morgan's laws) Let $\{A_\alpha\}_{\alpha\in I}$ be a collection of subsets of a larger set $\mathcal{B}$. I is simply an indexing set, that could be finite or infinite.

   (a) Show that
   $$\Big(\cup_{\alpha\in I} A_\alpha\Big)^c = \cap_{\alpha\in I} A_\alpha^c,$$
   where for any subset A, $A^c = \mathcal{B}\setminus A$ is the complement.

   (b) Show that
   $$\Big(\cap_{\alpha\in I} A_\alpha\Big)^c = \cup_{\alpha\in I} A_\alpha^c.$$

2. Decide which of the following statements are true and give a complete proof. For statements that are false provide a counter example.

   (a) If $A_1 \supseteq A_2 \cdots$ are all sets containing an infinite number of elements, then $\cap_{n=1}^{\infty} A_i$ is also an infinite set.

   (b) If $A_1 \supseteq A_2 \cdots$ are all finite non-empty sets of real numbers, then $\cap_{n=1}^{\infty} A_i$ is also finite and non-empty.

3. Use induction to prove the following.

   (a) $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

   (b) $1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$. **Note.** This is rather beautiful, that the sum of first n cubes is equal to the square of the sum of the first n numbers.

   (c) $11^n - 4^n$ is always divisible by 7.

   (d) $\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$.

4. Consider the inequality
   $$\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \leq 2.$$
   By part(d) above, this is clearly true. But now try to prove the inequality directly by induction. Why does it not work?