MATH 104 : Mid-Term

16 July, 2018

Name: ______________________________

- You have 100 minutes to answer the questions.
- Use of calculators or study materials including textbooks, notes etc. is not permitted.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.
- For questions with multiple parts, you can solve a part assuming the previous parts and get full credit for that particular part.

Question	Points	Score
1	18	
2	16	
3	12	
4	10	
5	14	
Total:	70	

1. (18 points) For each of the following, either give an example, or state that the request is impossible. If a request is impossible, provide a brief but compelling argument. For any example given, you do not need to prove that it has the required property.

   Unsolicited advice. Think each of these through carefully. Even if an answer pops out immediately, there is no harm in being careful.

   (a) A function f that is discontinuous at some p, but $\lim_{h\to 0}[f(p+h) - f(p-h)] = 0$.

   (b) An infinite bounded set $S \subset \mathbb{R}$ such that $\sup S$ is **<u>not</u>** a limit point of S.

   (c) A continuous, non-constant function $f : [a, b] \to \mathbb{R}$ such that the range $f([a, b])$ consists of only irrational numbers.

(d) A continuous function $f : [a, b] \to \mathbb{R}$ such that $f(t) > 0$ for all t, but $1/f$ is unbounded on $[a, b]$.

(e) A sequence $\{a_n\}$ with $\limsup_{n\to\infty} a_n = 1$ such that $a_n < 1$ for all n, and $a_n = 0$ for an infinite number of indices n.

(f) A sequence $\{a_n\}_{n=1}^{\infty}$ where $0 \leq a_n \leq 1/n$ for all n, but $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ diverges.

2. (a) (6 points) Given an $\varepsilon > 0$, find a $\delta > 0$ such that for all $x \in (1-\delta, 1+\delta)$,

$$\left|\frac{1}{2} - \frac{x}{1+x^2}\right| < \varepsilon.$$

(b) (7 points) Let $f(x) = |x|^3$. Show that f is differentiable on all of $\mathbb{R}$, and that $f'(x) = 3x|x|$.

(c) (3 points) Does $f''(0)$ exist? If so, compute it's value. If not, give a proper justification.

3. Let $f : (-1, 1) \to \mathbb{R}$ be defined by

$$f(x) = \frac{1}{\sqrt{1-x}}.$$

(a) (3 points) Write down the formulae for $f'(x)$, $f''(x)$ and $f^{(3)}(x)$.

(b) (6 points) Find a degree **two** polynomial $p(x) = ax^2 + bx + c$, such that for all $x \in [-1/2, 1/2]$,

$$|f(x) - p(x)| \leq \frac{5}{\sqrt{2}}|x|^3.$$

(please turn over for additional space to answer this part)

(cont.)

(c) (3 points) For the polynomial found in part (i) above, calculate

$$\lim_{x \to 0} \frac{\frac{1}{\sqrt{1-x}} - p(x)}{x^3},$$

if it exists (no justification needed), or prove that the limit does not exist.

4. (a) (5 points) Let $\alpha > 1$. Prove that for all $y > x > 1$,

$$\frac{y^\alpha - x^\alpha}{y - x} \geq \alpha x^{\alpha-1}.$$

(b) (5 points) Use the above part with $\alpha = 3/2$, to show that $f(x) = x\sqrt{x}$ is not uniformly continuous on $[1, \infty)$.

5. This final problem is about an algorithm to compute square roots. Let $x_1 > \sqrt{3}$ and define $x_2, x_3, \cdots$ recursively by

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{3}{x_n}\right)$$

(a) (2 points) Prove that $x_n > \sqrt{3}$ for all n.

(b) (3 points) Prove that $\{x_n\}$ is a decreasing sequence.

(c) (3 points) Prove that $\{x_n\}$ is convergent (quote the relevant theorem), and that $\lim_{n\to\infty} x_n = \sqrt{3}$.

Now, let $\varepsilon_n = x_n - \sqrt{3}$, that is, ε_n is the error in the approximation of $\sqrt{2}$ by x_n.

(d) (3 points) Show that

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{3}}.$$

(e) (3 points) Hence show that if $\beta = 2\sqrt{3}$, then

$$\varepsilon_{n+1} < \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^n}.$$

Remark. This shows that the algorithm is fantastically fast. For instance, if $x_1 = 2$ then already $\varepsilon_5 < 4 \cdot 10^{-16}$, so that the answer is correct up to 14 decimal places by just the fifth iteration of the algorithm!