3. SEQUENCES

• <u>Def^n</u>: A seq $\{x_n\}$ in (X, d) is said to converge to $p \in X$ if $\forall\ \varepsilon > 0$, $\exists\ N$ s.t.

$$n > N \implies d(x_n, p) < \varepsilon.$$

We say $\boxed{p = \lim_{n\to\infty} x_n}$.

If there is no such $p \in X$, we say $\{x_n\}$ <u>diverges</u>.

The set of points p_n is called the <u>range</u> of the seq.

$\{x_n\}$ is called bounded if the range is a bounded set.

<u>Rk</u>: If $Y \subset X$ with induced metric. A seq $\{x_n\}$ in Y might not converge in Y but might converge in X.

<u>Th^m</u> 3.1 Let $\{x_n\}$ seq in X.

(1). $\{x_n\}$ converges to p if and only if $\forall\ r > 0$, $B_r(p)$ contains all but finitely many terms in $\{x_n\}$.

(2) If $x_n \to p$ and $x_n \to p'$ Then $p = p'$.

(3) $\{x_n\}$ converges $\implies$ x_n is bounded.

(4) If $E \subset X$ and p is a l.p of $E \implies \exists\ \{x_n\}$ in E s.t. $x_n \to p$.

Pf 1) $\Longrightarrow$ $\forall\, r>0 \quad \exists\, N$ s.t $\quad d(x_n, p) < r \quad \forall\, n > N$

$\Rightarrow$ except for possibly $x_1, \dots, x_N$ all other $x_n \in B(p, r)$.

$\Longleftarrow$ $\forall\, r > 0$, all but finite term are in $B_r(p) \Rightarrow \exists\, N$ s.t $x_n \in B_r(p)\ \forall\, n > N$

$\Rightarrow \quad d(x_n, p) < r \quad \forall\, n > N$.

$\Rightarrow \quad x_n \to p$

2) Let $\varepsilon > 0$. Since $x_n \to p$ $\exists\, N_1$ s.t

$$d(p, x_n) < \varepsilon/2 \quad \forall\, n > N_1$$

$x_n \to p' \Rightarrow \exists\, N_2$ s.t

$$d(p', x_m) < \varepsilon/2 \quad \forall\, m > N_2.$$

If $N = \max(N_1, N_2) + 1 \Rightarrow$

$$d(p, x_N),\ d(p', x_N) < \varepsilon/2.$$

Δ-ineq $\Rightarrow \quad d(p, p') \le d(p, x_N) + d(p', x_N) \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$

So for any ε, $d(p, p') < \varepsilon \Rightarrow d(p, p') = 0$

$\Longrightarrow p = p'$

(c) Since $p_n \to p$; $\exists\ N$ s.t. $\forall\ n > N$, $d(p, p_n) < 1$.

Then let
$$R = \max(1, d(p,p_1), \ldots, d(p, p_N)).$$
Clearly $d(p, p_n) < R \ \forall\ n \implies \{p_n\}$ is bdd.

(d) Let $x_n \in B_{1/n}(p) \cap E$. Then $x_n \to p$.

- <u>Subsequences</u>.

<u>Def^n</u>: Given a sequence $\{x_n\}$ in X, and $\{n_k\}$ a sequence in $\mathbb{N}$, we call
$$\{x_{n_k}\}$$
a <u>subsequence</u> of $\{x_n\}$.

<u>Rk</u>: It is clear that $x_n \to p$ if and only if $x_{n_k} \to p$ for <u>every</u> subsequence $\{x_{n_k}\}$. Of course there are sequences s.t some subsequences converge. E.g. Let
$$x_n = \begin{cases} 1 & n \text{ odd} \\ -1 & n \text{ even} \end{cases}$$

If $n_k = 2k+1$, $x_{n_k} = 1 \ \forall\ k$. So $x_{n_k} \to 1$.

but x_n diverges.

<u>Th^m</u> 3.2.1) $\{x_n\}$ is a sequence in a compact metric space X, then $\exists\ \{n_k\}$ & $p \in X$ s.t.
$$x_{n_k} \to p$$

2) Every bounded seq in $\mathbb{R}^k$ has a convergent subsequence.

<u>Pf</u> 1) If $\{X_n\}$ is infinite $\Rightarrow$ $\exists\, p \in X$ a l.p.

Let $X_{n_k} \in B_{1/k}(p)$

Then $X_{n_k} \to p$.

If $E = \{X_n\}$ is finite. Then there is a $p \in E$ & $n_1 < n_2 \dots < n_k < \dots$ s.t

$$X_{n_k} = p \in X$$

Then $X_{n_k} \to p$.

2) If $\{x_n\}$ is bounded $\Rightarrow$ $X_n \in B_R(\vec{0})$ $\forall\, n$ and some $R > 0$. Then $\overline{B_R(\vec{0})}$ is compact $\Rightarrow$ there is a convergent sub-sequence by part (1).

<u>SEQUENCES IN $\mathbb{R}$</u>

• <u>Th^m</u> 3.3. Sps $\{s_n\}$ and $\{t_n\}$ are sequences in $\mathbb{R}$ and $s_n \to s$, $t_n \to t$. Then

(1) $\lim\limits_{n\to\infty} (s_n \pm t_n) = s \pm t$

(2) $\lim\limits_{n\to\infty} (c s_n) = cs$, $\lim\limits_{n\to\infty} (c + s_n) = c + s$ for any $c \in \mathbb{R}$

(3) $\lim\limits_{n\to\infty} s_n t_n = st$.

4) $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$, provided $s_n \neq 0$ for all $n \in \mathbb{N}$ and

5) (Squeeze) principle) If $r_n \le t_n \le s_n$, $r_n, s_n \to t \Rightarrow t_n \to t$.

<u>Pf</u>: We only prove 3) and 4). 1) and 2) are easier.

3) Note $\quad st - s_n t_n = st - s_n t + s_n t - s_n t_n$

$$= (s - s_n)t + (t - t_n)s_n$$

Let $\quad |s_n|, |t_n| < M \quad \forall n$. (since s_n, t_n converge they are bounded).

Given $\varepsilon > 0$

$s_n \to s \Rightarrow \exists N_1$ s.t

$$|s_n - s| < \frac{\varepsilon}{2M} \quad \forall n > N_1.$$

$t_n \to t \Rightarrow \exists N_2$ s.t

$$|t_n - t| < \frac{\varepsilon}{2M} \quad \forall n > N_2.$$

Let $N = \max(N_1, N_2)$. Then $\forall n > N$

$$|st - s_n t_n| < |s - s_n||t| + |t - t_n||s_n|$$

$$< \frac{\varepsilon}{2M} \cdot M + \frac{\varepsilon}{2M} \cdot M = \varepsilon.$$

So, $n > N \Rightarrow |st - s_n t_n| < \varepsilon \Rightarrow s_n t_n \to s \cdot t$.

4). Note $\quad \frac{1}{s} - \frac{1}{s_n} = \frac{s_n - s}{s_n s}$

Since $\quad s \neq 0 \quad \exists \quad \delta_1 > 0$ s.t $\quad |s_n| > \delta_1 \; \forall \; n > N$
& $|s| > \delta_1$.

Let $\delta = \min(\delta_1, |s_1|, |s_2|, \dots |s_N|)$.

Since $|s_n| \neq 0 \Rightarrow \delta > 0$.

Moreover $|s_n|, |s| > \delta$.

$s_n \to s \Rightarrow \exists N$ s.t. $\forall n > N$

$$|s_n - s| < \varepsilon \cdot \delta^2 .$$

Then $\forall n > N$.

$$\left| \frac{1}{s_n} - \frac{1}{s} \right| < \frac{|s_n - s|}{|s_n||s|} < \frac{\varepsilon \cdot \delta^2}{\delta^2} = \varepsilon$$

$$\Longrightarrow s_n^{-1} \longrightarrow s^{-1}.$$

- Sequence "toolkit" in $\mathbb{R}$.

Th^m 3.4 a) $p > 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 0$.

b) $p > 0$, then $\lim_{n\to\infty} \sqrt[n]{p} = 1$.

(c) $\lim_{n\to\infty} \sqrt[n]{n} = 1$

(d) $p > 0$ and $\alpha \in \mathbb{R}$, then

$$\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$$

(e) $|x| < 1$, then

$$\lim_{n\to\infty} x^n = 0$$

Rk (Real powers). For $\alpha = p/q$ and $x > 0$ we define

$$x^{p/q} = (x^{1/q})^p = (x^p)^{1/q}$$

If $\alpha \in \mathbb{R}$, we define x^α by the cut

We recall the binomial theorem

<u>Th</u>

Th^m 3.5 For $n \in \mathbb{N}$,

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

where $\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}$

Pf of Th^m 3.4: (a) For ~~so~~ $\varepsilon > 0$ simply pick $n > (1/\varepsilon)^{1/p}$.

(b). Sps $p \geq 1$, and let $x_n = p^{1/n} - 1$. Then $x_n \geq 0$.

Binomial theorem $\Rightarrow$

$$p = (1+x_n)^n \geq 1 + n x_n.$$

$$\Rightarrow \quad 0 \leq x_n \leq \frac{p-1}{n} \xrightarrow{n\to\infty} 0$$

$$\Rightarrow \quad x_n \to 0 \text{ as } n \to \infty \iff p^{1/n} \to 1.$$

If $p < 1$ then $1/p > 1$. So

$$\lim_{n\to\infty} \left(\frac{1}{p}\right)^{1/n} = 1 \Rightarrow \lim_{n\to\infty} p^{1/n} = 1.$$

(c). Again consider $x_n = \sqrt[n]{n} - 1 \geq 0$. By ~~B~~ binomial

$$n = (1+x_n)^n \geq \binom{n}{2} x_n^2 = \frac{n(n-1)}{2} x_n^2$$

$$\Rightarrow \qquad 0 \leq x_n \leq \sqrt{\frac{2}{n-1}} \xrightarrow{n\to\infty} 0$$

$$\Rightarrow \qquad x_n \longrightarrow 0 \Longrightarrow \lim_{n\to\infty} \sqrt[n]{n} = 1$$

(d). Let $k > \alpha$ be an integer. Then for $n > 2k$

$$(1+p)^n > \binom{n}{k} p^k = \frac{n(n-1)\cdots(n-k+1)}{k!} p^k$$

Since $n > 2k$, $\quad n - j \geq \frac{n}{2} \quad \forall\ j = 0, 1, \cdots k-1$

$$\Rightarrow \qquad (1+p)^n > \frac{n^k}{2^k \cdot k!} p^k$$

$$\Rightarrow \qquad 0 \leq \frac{n^\alpha}{(1+p)^n} \leq \frac{2^k \cdot k!}{p^k} \cdot n^{\alpha-k}$$

If $k > \alpha$ then $\alpha - k < 0 \Rightarrow n^{\alpha-k} \xrightarrow{n\to\infty} 0$.

$$\Rightarrow \qquad \lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$$

(e). Take $\alpha = 0$ and $p = \frac{1}{x} - 1$ in (d).

<u>Th</u>

Th^m 3.6 (Convergence in $\mathbb{R}^k$): If $\vec{x}_n = (x_{n,1}, \cdots, x_{n,k})$ is a sequence in $\mathbb{R}^k$. Then $\vec{x}_n \longrightarrow \vec{a} = (a_1, \cdots, a_k)$ if and only if $x_{n,j} \longrightarrow a_j \quad \forall\ j = 1, 2, \ldots, k$.

Pf: The proof follows from the inequality

$$\max_{j=1,\dots,k} |x_{n,j} - a_j| \leq |\vec{x}_n - \vec{a}| \leq \sqrt{k} \max_{j=1,\dots,k} |x_{n,j} - a_j|$$

$\Rightarrow$ Sps $\vec{x}_n \to \vec{a}$. Then $\forall\ \varepsilon > 0$, $\exists$ N s.t

$$n > N \implies |\vec{x}_n - \vec{a}| < \varepsilon.$$

But then $\max |x_{n,j} - a_j| < \varepsilon \quad \forall\ n > N$

$\Rightarrow$ for $j = 1, 2, \dots, k$

$$n > N \implies |x_{n,j} - a_j| < \varepsilon.$$

$$\Rightarrow \lim_{n\to\infty} x_{n,j} = a_j.$$

$\Leftarrow$ Conversely, sps $x_{n,j} \to a_j \ \forall\ j$. Then $\forall\ \varepsilon > 0\ \exists$ N_j s.t $|x_{n,j} - a_j| < \varepsilon/\sqrt{k} \quad \forall\ n > N_j$.

If $N = \max(N_1, \dots, N_k)$

$$n > N \implies |x_{n,j} - a_j| < \varepsilon/\sqrt{k} \quad \forall\ j$$

$$\Rightarrow \max_j |x_{n,j} - a_j| < \varepsilon/\sqrt{k}$$

$$\Rightarrow \quad |\vec{x}_n - \vec{a}| < \varepsilon$$

$$\Rightarrow \quad \lim_{n\to\infty} \vec{x}_n = \vec{a}$$

• Monotonic Sequences

Def^n: A sequence $\{S_n\}$ in $\mathbb{R}$ is said to be

(a) increasing if $S_{n+1} \geq S_n \ \forall n$, denoted by $S_n \uparrow$

(b) decreasing if $S_{n+1} \leq S_n \ \forall n$, denoted by $S_n \downarrow$

A sequence is called monotonic if it is increasing or decreasing

Th^m 3.7 Sps $\{S_n\}$ is monotonic. Then S_n converges if and only if it is bounded.

Rk: An increasing seq (resp decreasing) is always bounded below (resp. above) by S_1.

Pf: Sps $\{S_n\}$ is increasing. The other case is similar.

$\Rightarrow$ True for general converging sequences.

$\Leftarrow$ Sps $\{S_n\}$ is bounded. Let

$$S = \sup S_n.$$

Clearly $S_n \leq S \ \forall n$. Let $\varepsilon > 0$. Then $S - \varepsilon$ is not an upper bound for S_n. So $\exists N$ s.t.

$$S - \varepsilon \leq S_N.$$

Since S_n is increasing, $\forall n > N$.

$$S - \varepsilon < S_n$$

$\Rightarrow \forall n > N, \quad S - \varepsilon \leq S_n \leq S$ or $|S - S_n| < \varepsilon$.

$\Rightarrow \quad s_n \to s$.

• Rk: In fact the proof shows

$s_n \uparrow$ and bounded $\Rightarrow \lim_{n\to\infty} s_n = \sup s_n$

$s_n \downarrow$ and bdd. $\Rightarrow \lim_{n\to\infty} s_n = \inf s_n$

• Extended real numbers

Def^n 1) We say $\{s_n\}$ converges to $+\infty$ if $\forall\, M > 0$ $\exists\, N$ s.t

$n > N \Rightarrow s_n > M$.

2) We say $s_n \to -\infty$ if $\forall\, M > 0$, $\exists\, N$ s.t

$n > N \Rightarrow s_n < -M$.

Def^n: We define the extended real line by

$$\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \cup \{-\infty\}.$$

Th^m 3.7*: Sps $\{s_n\}$ is monotonic. Then s_n converges in $\hat{\mathbb{R}}$.

So if $s_n \uparrow$ then $s_n \to s \in \mathbb{R} \Leftrightarrow \{s_n\}$ is bdd.

s_n unbounded $\Rightarrow s_n \to +\infty$.

III^{ly} $s_n \downarrow$ then $s_n \to -\infty$ if s_n is unbounded.

· Limit Superior (limsup) and Limit Inferior

Let $\{a_n\}$ be a seq. in $\mathbb{R}$ which is upper bounded.

Then if
$$u_n = \sup\{a_k \mid k \geq n\},$$

clearly $u_{n+1} \leq u_n$ (since sup is taken over a smaller set), so $u_n \downarrow$,

Similarly we define
$$l_n = \inf\{a_k \mid k \geq n\}.$$

Then $l_n \uparrow$.

Def^n: 1) The limit superior of a seq $\{a_n\}$ is defined to be
$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} u_n = \inf \sup\{a_k \mid k \geq n\}.$$

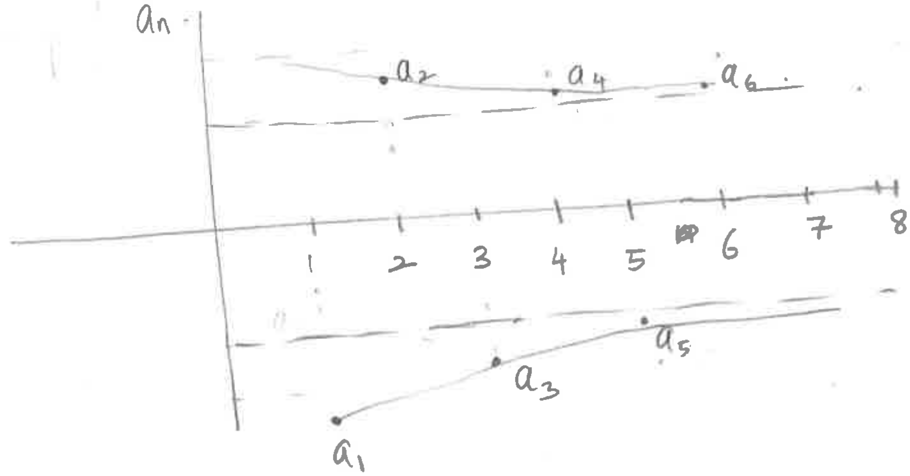
if $\{a_n\}$ bdd. above or $+\infty$ otherwise.

2) The limit inferior of a seq $\{a_n\}$ is defined to be
$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} l_n = \sup \inf\{a_k \mid k \geq n\}.$$

if a_n is bdd. below or $-\infty$ otherwise.

Example: Consider $a_n = (-1)^n(n+1)/n$.

$$u_n = \sup\{(-1)^k(k+1)/k \mid k \geq n\}.$$



Clearly $u_2 = a_2,\ u_3 = u_4 = a_4,\ u_5 = u_6 = a_6 \ldots$

$\Rightarrow \quad \inf u_n = 1$

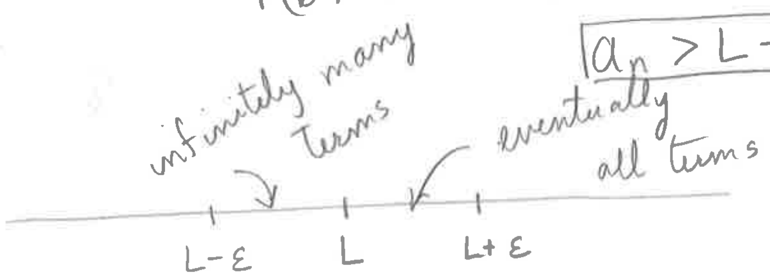
$\Rightarrow \quad \limsup_{n\to\infty} a_n = 1.$

Similarly $\liminf_{n\to\infty} a_n = -1.$

Th^m 3.8 For a seq $\{a_n\}$ in $\mathbb{R}$,

1) $L = \limsup_{n\to\infty} a_n \iff$ 1(a) $\forall \varepsilon > 0,\ \exists N$ s.t $\boxed{a_n \leq L+\varepsilon}$ $\forall n > N$.

1(b) $\forall \varepsilon > 0,\ \forall N,\ \exists\ n > N$ s.t $\boxed{a_n > L - \varepsilon}$



2) $l = \liminf_{n\to\infty} a_n \iff$ 2(a) $\forall \varepsilon > 0,\ \exists N$ s.t

$a_n > l - \varepsilon \quad \forall n > N.$

2(b) $\forall \varepsilon > 0,\ \forall N > 0\ \exists\ n > N$ s.t

$a_n < l + \varepsilon.$

$\underline{\text{Pf}}$ (1) Sps $L < +\infty$. Let $u_n = \sup\{a_k \mid k \geq 0\}$

$\Rightarrow$ Let $\varepsilon > 0$

Since $L = \inf u_n \Rightarrow \exists\, N$ s.t $u_N < L + \varepsilon$.

$\Rightarrow \forall\, n > N \quad a_n < L + \varepsilon$.

Also $u_N > L - \varepsilon \quad \forall\, N$ since $L = \inf u_n$.

$\Rightarrow$ for any N, $\exists\, n > N$ s.t

$$a_n > L - \varepsilon$$

or else $\sup\{a_n \mid n \geq N\} \leq L - \varepsilon$ contradicting $u_N > L - \varepsilon$.

Done!

$\Leftarrow$ Sps L satisfies properties (a) & (b), let $U = \limsup_{n\to\infty} a_n$

For any $\varepsilon > 0$, $\exists\, N$ s.t

$$a_n < L + \varepsilon \quad \forall\, n > N$$

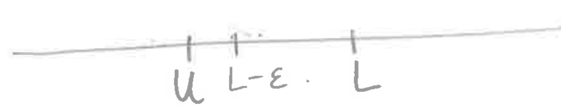
$$\Rightarrow \quad u_N < L + \varepsilon.$$

$$\Rightarrow \quad U = \inf u_n < L + \varepsilon.$$

So $U < L + \varepsilon \quad \forall\, \varepsilon > 0 \Rightarrow U \leq L$.

$\underline{\text{Claim}}$: $U = L$.

$\underline{\text{Pf}}$: Sps $U < L$. Choose $\varepsilon > 0$ s.t $U < L - \varepsilon < L$.



For any N, $\exists$ $n > N$ s.t

$$a_n > L - \varepsilon$$

$$\Rightarrow \quad u_N > L - \varepsilon \quad \forall\ N$$

$$\Rightarrow \quad u > L - \varepsilon > u \quad \text{contradiction!}$$

So $L = u$.

Similar argument works if $L = +\infty$ or for part (2).

<u>Th</u>^{m} 3.9 Let $\{a_n\}$ seq in $\mathbb{R}$. Then

1) $$\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n.$$

2) $a_n \to L$ if and only if

$$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = L.$$

<u>Pf</u>: 1). If $L = \limsup\limits_{n\to\infty} a_n < \liminf\limits_{n\to\infty} a_n$

Choose ε s.t $L + \varepsilon < \liminf\limits_{n\to\infty} a_n$.

1(a) $\Rightarrow$ $a_n < L + \varepsilon \quad \forall\ n > N$, for some N

$\Rightarrow$ $l_N < L + \varepsilon \quad \forall\ N$.

$\Rightarrow$ $\liminf a_n < L + \varepsilon$ contradiction

2) $\Rightarrow$. $\forall\ \varepsilon > 0$, $\exists$ N s.t

$$n > N \Rightarrow \quad |a_n - L| < \varepsilon$$

$$\Leftrightarrow \quad L - \varepsilon < a_n < L + \varepsilon$$

So 1(a),(b) & 2(a),(b) can be verified

So $L = \limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$.

$\Leftarrow$ $\forall\ \varepsilon > 0$, $\limsup a_n = L$

$\Rightarrow \exists\ N_1$ s.t $n > N_1 \Rightarrow a_n < L + \varepsilon$.

$\liminf a_n = L \Rightarrow \exists\ N_2$ s.t

$n > N_2 \Rightarrow a_n > L - \varepsilon$.

If $N = \max(N_1, N_2)$. Then

$n > N \Rightarrow L - \varepsilon < a_n < L + \varepsilon$

$\Leftrightarrow |a_n - L| < \varepsilon$.

$\Rightarrow a_n \to L$.

CAUCHY SEQUENCES & COMPLETENESS

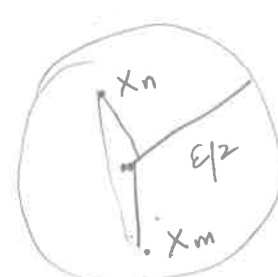
• Def^n A sequence $\{x_n\}$ in (X, d) is called Cauchy if $\forall\ \varepsilon > 0\ \exists\ N$ s.t $\forall\ n, m > N$

$$d(x_n, x_m) < \varepsilon.$$

Th^m 3.10 If $\{x_n\}$ converges, then it is Cauchy.

Pf: Sps $x_n \to p$. Let $\varepsilon > 0$. Then $\exists\ N$ s.t

$\forall\ n > N$ we have $d(p, x_n) < \varepsilon/2$.



But then if $n, m > N$

Δ-ineq $\Rightarrow$ $d(x_n, x_m) \leq d(p, x_n) + d(p, x_m)$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Done!

Ques: What about converse? Does every Cauchy seq. converge?

Example Consider $\mathbb{Q}$ with usual metric. For n, let $x_n \in (\sqrt{2} - 1/n, \sqrt{2}) \cap \mathbb{Q}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can always choose such an x_n. Clearly $\{x_n\}$ is Cauchy since if $n < m$,

$$|x_n - x_m| < \frac{1}{n} - \frac{1}{m} < \frac{1}{n}.$$

So if $\varepsilon > 0$ & $N > 1/\varepsilon$ Then $\forall n, m > N$

$$|x_n - x_m| < \frac{1}{n} < \frac{1}{N} < \varepsilon.$$

But $\{x_n\}$ does not converge in $\mathbb{Q}$.

Def^n A metric space (X, d) is called complete if every Cauchy sequence converges

Th^m 3.11 (1) A compact metric space is complete

(2) $\mathbb{R}^k$ is complete

Pf: 1) Let (X, d) be compact and $\{x_n\}$ a Cauchy seq.

Then $\exists$ sub-seq x_{n_k} & $p \in X$ st $x_{n_k} \to p$.

Claim $x_n \to p$

Pf Let $\varepsilon > 0$. Cauchy $\Rightarrow$ $\exists$ N s.t $\forall$ $n, m > N$

$$d(x_n, x_m) < \frac{\varepsilon}{2}.$$

Since $x_{n_k} \to p$, $\exists$ k s.t $n_k > N$ and

$$d(p, x_{n_k}) < \frac{\varepsilon}{2}.$$

Then for any $n > N$,

$$d(p, x_n) \leq d(p, x_{n_k}) + d(x_{n_k}, x_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\Rightarrow$ $x_n \to p$

2) Sps $\{x_n\}$ in $\mathbb{R}^k$ is Cauchy.

Claim $\{x_n\}$ is bounded.

Pf: $\exists$ N s.t $\forall$ $n > N$, $d(x_n, x_N) < 1$.

Let $R = \max(1, d(x_1, x_N), \ldots, d(x_{N-1}, x_N))$

Clearly $x_n \in B_R(x_N)$ $\forall$ N

$\Rightarrow$ $\{x_n\}$ is bounded

But then $\{x_n\}$ is contained in a compact set $(\overline{B_R(x_N)})$ & hence converges by part 1).

Rk: So to prove convergence in $\mathbb{R}^k$ we only need to check if seq is Cauchy.