8. Power Series

- Given a seq^n $\{c_n\}$ of real numbers, the series

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

is called a <u>power series</u>. The numbers c_n are called <u>coefficients</u>, and a is called the center.

<u>Th^m</u> 8.1 (Fundamental Theorem of power serie) Given a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$.

1) The series converges <u>absolutely</u> on $|x-a| < R$ and diverges on $|x-a| > R$, where

$$R = \frac{1}{\alpha}, \quad \alpha = \limsup_{n\to\infty} |c_n|^{1/n}.$$

R is called <u>radius</u> of convergence.

2) Convergence is uniform on $|x-a| \le R - \varepsilon \ \forall \ \varepsilon > 0$

3) For $|x-a| < R$, if we denote $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$, then f is cont. & diff. on $(-R, R)$. Moreover,

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

For uniform convergence, we will need the foll

<u>Th^m</u> 8.2 (Weierstrass M-test) Sps $\{f_n\}$ is a seq of functions, $f_n : E \to \mathbb{R}$ s.t

$$|f_n(x)| \le M_n \quad \forall\ x \in E,\ \forall\ n$$

If $\sum M_n$ conv. $\Rightarrow$ $\sum f_n$ conv. <u>uniformly</u> on E.

<u>Pf</u>: $\sum M_n$ conv $\Rightarrow$ $\forall\ \varepsilon$, $\exists\ N$ s.t $\forall\ m, n > N$

$$\sum_{k=n}^{m} M_k < \varepsilon.$$

$$\Rightarrow \quad \sum_{k=n}^{m} |f_k(x)| < \sum_{k=n}^{m} M_k < \varepsilon.$$

$\Rightarrow$ partial sums of $\sum f_n$ are uniformly Cauchy $\Rightarrow$ $\sum f_n$ conv. uniformly.

<u>Pf of Th^m 8.1</u> Without loss of generality, sps $\boxed{a = 0.}$

1) Let $a_n = c_n x^n$.

$$\limsup_{n\to\infty} |a_n|^{1/n} = \limsup_{n\to\infty} |c_n x^n|^{1/n}$$

$$= |x| \limsup_{n\to\infty} |c_n|^{1/n} = |x| \cdot \alpha$$

Root test $\Longrightarrow$ $\sum c_n x^n$ conv. absolutely if

$|x| < 1/\alpha = R$.

diverges if $|x| > 1/\alpha = R$.

2) Let $\varepsilon > 0$. On $|x| \leq R - \varepsilon$

$$|c_n x^n| \leq c_n (R - \varepsilon)^n.$$

Again if $a_n = c_n (R-\varepsilon)^n$, by root test

$\sum_{n=0}^{\infty} c_n (R-\varepsilon)^n$ converges.

M-test $\Longrightarrow$ on $|x| \leq R - \varepsilon$, $\sum_{n=0}^{\infty} c_n x^n$ converges uniformly

3) Since $c_n x^n$ is cont., conv. is uniform

$\Longrightarrow f(x) = \sum_{n=0}^{\infty} c_n x^n$ is cont

To compute derivative let

$$S_N = \sum_{n=0}^{N} c_n x^n$$

be the N^{th} partial sum. on $(-R, R)$.

Since S_n is a polynomial, S_n is diff.

&
$$S_N' = \sum_{n=1}^{N} n \cdot c_n x^{n-1}$$

Now, S_N' are partial sums of
$$T(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

<u>Claim</u>: T also has radius of conv. R and conv is uniform on $|x| \leq R - \varepsilon$ $\forall$ $\varepsilon > 0$.

Assuming this, $\Rightarrow$ if $p \in (-R, R)$, let $\varepsilon > 0$ s.t. $p \in (-R+\varepsilon, R-\varepsilon)$. Since $\{S_N'\}$ conv uniformly on $[-R+\varepsilon, R-\varepsilon]$ and $S_N(p)$ converge by Th^m on uniform conv. & diff.

$\Rightarrow$ $\{S_N\}$ converges to some S &
$$S'(p) = \lim_{N \to \infty} S_N'(p).$$

But we know $S_N \xrightarrow{u.c} f$. So
$$f'(p) = \lim_{N \to \infty} S_N'(p) = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

<u>Pf of Claim</u>: Let $b_n = n c_n x^{n-1}$.

$$\limsup_{n \to \infty} |b_n|^{1/n} = \limsup_{n \to \infty} n^{1/n} |c_n|^{1/n} |x|^{1-1/n}$$

But $n^{1/n} \xrightarrow{n\to\infty} 1$, $|x|^{1-1/n} \longrightarrow |x|$.

So $\limsup_{n\to\infty} |b_n|^{1/n} = |x| \limsup_{n\to\infty} |C_n|^{1/n} = |x| \cdot \alpha$.

We need this < 1 i.e $|x| < 1/\alpha = R$.

<u>Def^n</u>: R is called radius of convergence.

The set of x s.t the power series converges is called the <u>interval of convergence</u>.

<u>Example</u>: 1) $\sum_{n=0}^{\infty} x^n$ conv if $|x| < 1$

$R = 1$ In fact

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x| < 1$$

If $x = \pm 1$, series diverges. So $I = (-1, 1)$.

2) $\sum \frac{x^n}{n}$, $C_n = 1/n$.

$$\alpha = \limsup_{n\to\infty} |C_n|^{1/n} = \limsup_{n\to\infty} \frac{1}{n^{1/n}} = 1$$

$\Rightarrow \boxed{R = 1}$

At $x = 1$, $\sum \frac{1}{n}$ diverges
$x = -1$, $\sum \frac{(-1)^n}{n}$ conv.

$\Rightarrow I = [-1, 1)$

3) $\sum_{n=0}^{\infty} \frac{x^n}{n^2}$, $C_n = 1/n^2$.

$$R = \frac{1}{\limsup_{n\to\infty} |C_n|^{1/n}} = \frac{1}{\limsup_{n\to\infty} n^{-2/n}} = 1$$

$$\left.\begin{aligned} x = -1, &\quad \sum \frac{(-1)^n}{n^2} \text{ conv.} \\ x = 1, &\quad \sum \frac{1}{n^2} \text{ conv} \end{aligned}\right\} \Rightarrow I = [-1, 1]$$

<u>Cor 8.3</u>: Let $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ be a power series with radius of convergence R. Then f has derivatives of all orders in $|x-a| < R$. Moreover

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) c_n x^{n-k}.$$

In particular

$$\boxed{c_n = \frac{f^{(n)}(a)}{n!}}$$

<u>Pf</u>: In Th^m 8.1 we showed that f' exist and

$$f'(x) = \sum_{n=1}^{\infty} n \cdot c_n (x-a)^{n-1}$$

We also showed in the proof that the R.H.S is a power series with rad. of conv. R. Now apply the argument.

repeatedly to conclude that f has derivatives of all orders &

$$f^{(k)}(x) = \sum_{n=k}^{\infty} c_n n(n-1) \cdots (n-k+1)(x-a)^{n-k}$$

Putting $x = a$,

$$f^{(k)}(a) = k(k-1) \cdots 1 \cdot c_k$$

$$c_k = \frac{f^{(k)}(a)}{k!}$$

Cor. 8.4: Let $f = \sum_{n=0}^{\infty} c_n (x-a)^n$ have rad. of convergence $R > 0$. Then for any $a-R < c < d < a+R$ $f \in R[c, d]$, and moreover

$$\int_c^d f(x)\,dx = \sum_{n=0}^{\infty} c_n \int_c^d (x-a)^n\,dx$$

Example: For $|x| < 1$,

$$\frac{1}{1+t} = \frac{1}{1-(-t)} = 1 - t + t^2 - t^3 + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n t^n$$

Integrating

$$\ln(1+x) = \int_0^x \frac{dt}{1+t} = \sum_{n=0}^{\infty} (-1)^n \int_0^x t^n\,dt :$$

$$= \sum_{n=0}^{\infty} (-1)^n \left. \frac{t^{n+1}}{n+1} \right|_0^x$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

$$\Rightarrow \ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

for $|x| < 1$.

Note that the original series diverged at $x = \pm 1$. The integrated series though, converges at $x = -1$. Also $\ln(1+x) = \ln 2$ at $x = -1$.

<u>Ques</u>: Is it true that

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} \cdots \quad ?$$

<u>In contrast</u>: $\frac{1}{1-x} = 1 + x + x^2 + \cdots$ for $|x| < 1$. Series diverges at $x = -1$. but the fraction equals a sensible value, namely $1/2$.

<u>Th^m</u> (Abel) Suppose $\sum_{n=0}^{\infty} c_n r^n$ converges. Then $f(x) = \sum_{n=0}^{\infty} c_n x^n$ has radius of convergence $R \geq r$. Moreover

$$\lim_{x \to r^-} f(x) = \sum_{n=0}^{\infty} c_n r^n.$$

Pf: Let $r = 1$.

Let $S_n = c_0 + c_1 + \cdots + c_n$, $S_{-1} = 0$

Then

$$\sum_{n=0}^{m} c_n x^n = \sum_{n=0}^{m} (S_n - S_{n-1}) x^n$$

$$= \sum_{n=0}^{m-1} S_n x^n - \sum_{n=0}^{m} S_{n-1} x^n + S_m x^m$$

$$= \sum_{n=0}^{m-1} S_n x^n - \sum_{n=0}^{m-1} S_n x^{n+1} + S_m x^m$$

$$= (1-x) \sum_{n=0}^{m-1} S_n x^n + S_m x^m$$

For $|x| < 1$, let $m \to \infty$. Then

$$f(x) = (1-x) \sum_{n=0}^{\infty} S_n x^n$$

Sps $s = \lim_{n\to\infty} S_n = \sum c_n$. Given $\varepsilon > 0$, $\exists N$

s.t $\forall n > N$,

$$|s - S_n| < \frac{\varepsilon}{2}.$$

Note that for $|x| < 1$, $(1-x) \sum_{n=0}^{\infty} x^n = 1$.

$$\Rightarrow \quad f(x) - s = (1-x) \sum_{n=0}^{\infty} S_n x^n - s(1-x) \sum_{n=0}^{\infty} x^n$$

$$= (1-x) \sum_{n=0}^{\infty} (S_n - s) x^n$$

So $|f(x) - s| \leq \left| (1-x) \sum_{n=0}^{\infty} (S_n - s) x^n \right|$

$$\le (1-x)\sum_{n=0}^{N}|S_n - s||x|^n + (1-x)\sum_{n=N+1}^{\infty}|S_n - s||x|^n$$

$$= I_1 + I_2$$

$$I_2 \le (1-x)\frac{\varepsilon}{2}\sum_{n=N+1}^{\infty} x^n \le \frac{\varepsilon}{2}\left(\sum_{n=0}^{\infty} x^n\right)(1-x) = \frac{\varepsilon}{2}.$$

$$I_1 \le (1-x)\max(S_n)\cdot N.$$

If $x > 1-\delta$, where $\delta = \dfrac{\varepsilon}{2N\max(S_n)}$.

$$\Rightarrow \quad I_1 \le \frac{\varepsilon}{2}$$

So for $x > 1-\delta$,

$$|f(x) - s| < \varepsilon \Rightarrow \lim_{x\to 1^-} f(x) = s$$

<u>Example</u> Let $c_n = \dfrac{(-1)^{n-1}}{n}$. Then $\sum c_n$ converges.

Put $f(x) = \ln(1+x)$. Then on $|x|<1$

$$\ln(1+x) = \sum_{n=1}^{\infty} c_n x^n$$

$$\text{Th}^m \rightarrow \lim_{x\to 1^-}\ln(1+x) = \sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{n}$$

$$\shortparallel$$

$$\ln 2$$

$$\text{So} \quad \ln 2 = \sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{n}$$

• Question: If $f \in C^\infty(a-R, a+R)$ can f be written as a power series.

Def^n: The Taylor series of f centered at 'a' is the power series

$$T_f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

The n^{th} Taylor polynomial is defined as

$$T_{f,n}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Ques: When is $f(x) = T_f(x)$ on some interval $(a-\varepsilon, a+\varepsilon)$, $\varepsilon > 0$?

Example:
$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Claim: $f \in C^\infty(\mathbb{R})$, $f^{(k)}(0) = 0 \ \forall k$.

Pf: $k=1$,

$$f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{e^{-1/h^2}}{h}$$

$$= \lim_{t\to\infty} t \cdot e^{-t^2} = 0$$

For $x \neq 0$, $\quad f'(x) = \frac{2}{x^3} \cdot e^{-1/x^2}$

Sub claim: For all k, $\exists$ polynomial P_k s.t. for $x \neq 0$,

$$f^{(k)}(x) = P_k\left(\frac{1}{x}\right) \cdot e^{-1/x^2}.$$

Pf: True for $k = 1$, $P_1(t) = 2t^3$.

Assume for $k = n$ i.e. $f^{(n)}(x) = P_n\left(\frac{1}{x}\right) \cdot e^{-1/x^2}$, $x \neq 0$.

Then $f^{(n+1)}(x) = P_n'\left(\frac{1}{x}\right) \cdot \left(\frac{-1}{x^2}\right) \cdot e^{-1/x^2} + P_n\left(\frac{1}{x}\right) \cdot \frac{2}{x^3} \cdot e^{-1/x^2}$.

$$= e^{-1/x^2}\left[-P_n'(1/x) \cdot \left(\frac{1}{x^2}\right) + \frac{2}{x^3} P_n\left(\frac{1}{x}\right)\right]$$

Let $P_{n+1}(t) = 2t^3 P_n(t) - t^2 P_n'(t)$.

Done!

So now, again by induction if $f^{(n)}(0) = 0$.

$$f^{(n+1)}(0) = \lim_{h \to 0} \frac{f^{(n)}(h)}{h} = \lim_{h \to 0} \frac{P_n(1/h) \cdot e^{-1/h^2}}{h}.$$

$$= \lim_{t \to \infty} t P_n(t) \cdot e^{-t^2} = 0.$$

Since e^{t^2} grows faster than any polynomial.

$\underline{Th^m}$: Let $f \in C^\infty[c,d]$, and, $a \in (c,d)$. Sps $\exists\ \varepsilon > 0$ and const M (might depend on a, ε) s.t

$$|f^{(n)}(x)| \leq M^n$$

$\forall n$, $\forall x \in (a-\varepsilon, a+\varepsilon)$. Then for each $x \in (a-\varepsilon, a+\varepsilon)$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

$\underline{Pf}$: By Taylor's theorem, $\exists\ c$ between x and a s.t

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n)}(c)}{n!}(x-a)^n.$$

$$= T_{n-1}(x) + E_n(x).$$

$\underline{\text{Claim}}$: $\lim_{n\to\infty} E_n(x) = 0$.

$$|E_n(x)| \leq \frac{|f^{(n)}(c)|\,|x-a|^n}{n!} \leq \frac{M^n \varepsilon^n}{n!} = \frac{(\varepsilon M)^n}{n!}$$

But $\lim_{n\to\infty} \frac{p^n}{n!} = 0$ for any $p \in \mathbb{R}$ since

$e^p = \sum p^n/n!$ is a convergent series

$$\Rightarrow \lim_{n\to\infty} |E_n(x)| = 0.$$

Example $f(x) = \ln x$ on $(0, \infty)$

$a = 1$.

$$f'(x) = \frac{1}{x}, \quad f''(x) = -\frac{1}{x^2}, \quad f'''(x) = 2x^{-3}$$

$$f^{(4)}(x) = -6x^{-4} \qquad f^{(n)}(x) = (-1)^{n-1} \cdot (n-1)!\, x^{-n}.$$

If $x > 1-\varepsilon$, $|f^{(n)}(x)| < \frac{(n-1)!}{(1-\varepsilon)^n}$

Not quite poly M^n. So cannot use theorem directly. By proof of th^m

$$|E_n(x)| \le \frac{|f^{(n)}(c)|}{n!} |x-1|^n$$

Sps $|x-1| < \varepsilon$, $x > 1$, then $c > 1$ &.

$$|f^{(n)}(c)| \le (n-1)!$$

$$|E_n(x)| \le \frac{1}{n} \cdot |x-1|^n \longrightarrow 0 \quad \text{if } |x-1| < 1.$$

Sps $x < 1$, $|x-1| < \varepsilon$, then

$$|f^{(n)}(c)| = (n-1)!\, c^{-n} \le (n-1)!\, x^{-n}$$

$$|E_n(x)| \le \frac{(n-1)!}{x^n} \cdot \frac{(1-x)^n}{n!} = \frac{(1-x)^n}{x^n} \cdot \frac{1}{n}$$