5. Limits & Continuity

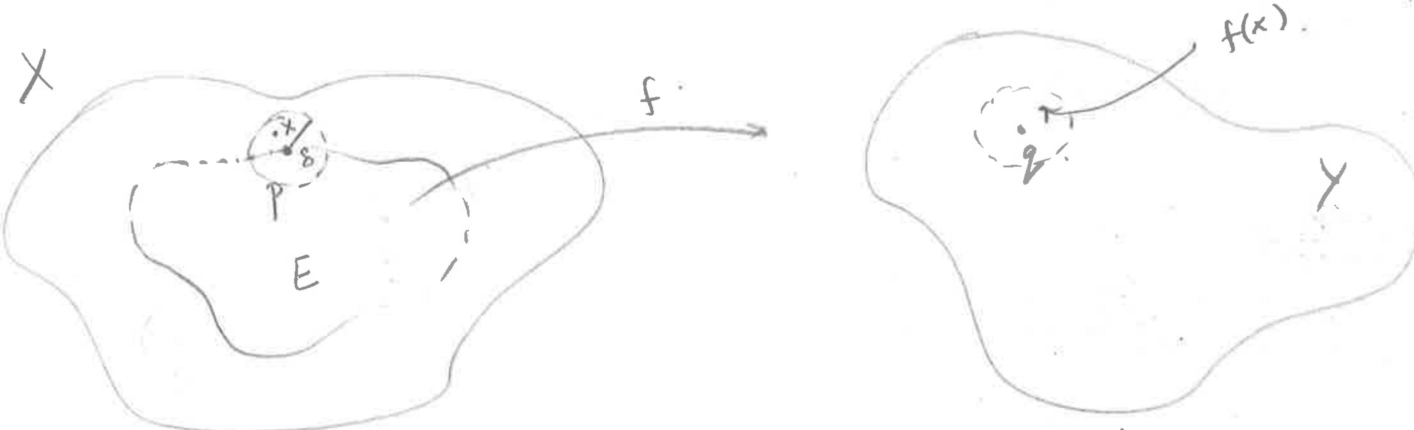
• Limits

Def^n Let (X, d_X) and (Y, d_Y) be metric spaces. Let $E \subset X$ and $f: E \to Y$ a function. For $p \in X$ a l.p of E, we say $f(x) \to q$ as $x \to p$ or denote

$$\lim_{x \to p} f(x) = q.$$

if $\forall\ \varepsilon > 0$, $\exists\ \delta = \delta(\varepsilon)$ s.t

$$d_X(p, x) < \delta \implies d_Y(f(x), q) < \varepsilon.$$



Rk: $f(x)$ need not be defined at $x = p$. Moreover, even if $p \in E$, we might not have that $f(p)$ equals $\lim_{x \to p} f(x)$.

Example: 1) Consider $f: \mathbb{R} \to \mathbb{R}$ defined by

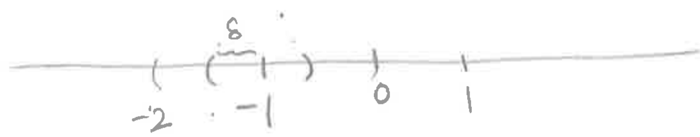
$$f(x) = x^2.$$

Claim: $\lim_{x \to -1} f(x) = 1$.

Pf: Given $\varepsilon > 0$.

Aim: $\exists\ \delta > 0$ s.t $|x+1| < \delta \Rightarrow |x^2-1| < \varepsilon$.

$|x^2-1| = |x-1||x+1|$



Note that if $\delta < 1$. Then $|x+1| < \delta \Rightarrow |x-1| < 3$.

$\Rightarrow \quad |x^2-1| < 3|x+1| < 3\delta$.

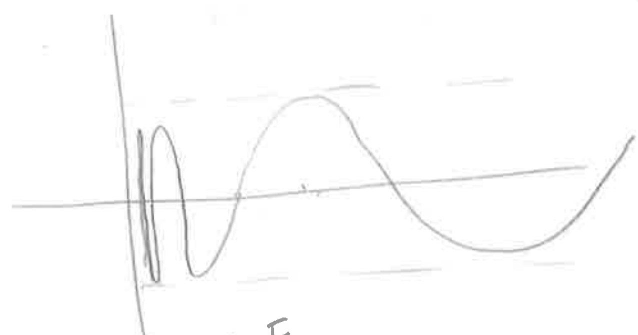
We want this to be smaller than ε, so pick

$$\boxed{\delta = \min\left(\frac{\varepsilon}{3}, 1\right)}$$

More generally, one can show $\lim_{x \to p} x^2 = p^2$.

2). Consider $f : (0,\infty) \to \mathbb{R}$, $f(x) = \sin(1/x)$.

Then $\lim_{x \to 0} f(x)$ does not exist.



Thm 5.1

1) $\lim_{x \to p} f(x) = q \iff \forall \{p_n\}$ in E, $p_n \neq p\ \forall n$, $p_n \to p$ s.t. $\lim_{n \to \infty} f(p_n) = q$.

2) Limits, if they exist, are unique.

Pf 1) $\Longrightarrow$ Let $\varepsilon > 0$, $\exists\ \delta$ s.t

$$d_X(x, p) < \delta \Rightarrow d_Y(f(x), q) < \varepsilon.$$

If $p_n \to p$, then $\exists\ N$ s.t $\forall\ n > N$

$$d_X(p_n, p) < \delta$$

$$\Rightarrow \quad d_Y(f(p_n), q) < \varepsilon.$$

So $\forall\ \varepsilon > 0$, $\exists\ N$ s.t $\forall n > N$, $d_Y(f(p_n), q) < \varepsilon$.

$$\Rightarrow \quad f(p_n) \to q.$$

$\Longleftarrow$ Proceed by contradiction. Sps. $\lim_{x\to p} f(x) \neq q$.

Then $\exists\ \varepsilon > 0$. s.t $\forall\ \delta$, $\exists$ a point $x_\delta \in B^X_\delta(p) \cap E$

s.t $\quad d_Y(f(x_\delta), q) > \varepsilon$.

Take $\delta = 1/n$. Since p is a l.p. $\exists\ p_n \in B_{1/n}(p) \cap E$ s.t

$p_n \neq p$. Then $d_Y(f(p_n), q) > \varepsilon$.

So $\{p_n\}$ is a seq in E s.t $\lim_{n\to\infty} f(p_n) \neq q$.

which is a contradiction.

(2) Follows from (1) & similar property that we proved for sequences.

Th^m 5.2 Let $E \subset X$ and p a l.p.

1) If $f, g : E \to \mathbb{C}$ s.t

$$\lim_{x\to p} f(x) = A, \quad \lim_{x \to p} g(x) = B.$$

Then

(a) $\lim_{x \to p} (f+g)(x) = A+B$.

(b) $\lim_{x \to p} (fg)(x) = AB$.

(c) $\lim_{x \to p} \left(\frac{f}{g}\right)(x) = \frac{A}{B}$, if $B \neq 0$.

2) If $\vec{F}, \vec{G} : E \to \mathbb{R}^k$ with $\vec{F} = (F_1, \dots, F_k)$. $\vec{G} = (G_1, \dots, G_k)$

Then (a) holds. Moreover we have

(b') $\lim_{x \to p} (\vec{F} \cdot \vec{G})(x) = \vec{A} \cdot \vec{B}$

Rk: $(f+g)(x) := f(x) + g(x)$

$(\vec{F} \cdot \vec{G})(x) := \sum_{j=1}^{k} F_j(x) G_j(x)$.

· Continuous functions

Defn $(X, d_X), (Y, d_Y)$ metric spaces, $E \subset X$ and $f : E \to Y$

We say f is continuous at $p \in E$ if $\forall \varepsilon > 0, \exists \delta$

s.t $d_X(x, p) < \delta \implies d_Y(f(x), f(p)) < \varepsilon$.

We say f is cont on E if it is cont. at all $p \in E$.

Thm 5.3 If p is a l.p of E, then f is cont at p

$\iff \lim_{x \to p} f(x) = f(p)$.

A function not cont. at p is called discontinuous at p.

Rk 1) If $E \subset \mathbb{R}^k$, an equivalent definition of cont. at $\vec{p} \in E$ is: $\forall\, \varepsilon > 0$, $\exists\, \delta$ s.t

$$|\vec{h}| < \delta \implies |f(\vec{p} + \vec{h}) - f(\vec{p})| < \varepsilon$$

2) If $p \in E$ is an isolated point, f is automatically cont.

Thm 4.4: Let $E \subset X$.

1) If $f, g : E \to \mathbb{C}$ are cont. at p, then so are $f \pm g$ & fg. If $g(p) \neq 0$, f/g is also cont. at p

2) $\vec{F}, \vec{G} : E \to \mathbb{R}^k$, $\vec{F} = (F_1, \dots, F_k)$. $\vec{G} = (G_1, \dots, G_k)$.

(a) $\vec{F}$ cont at $p \Leftrightarrow F_j$ cont at p $\forall\, j$.

(b) $\vec{F}, \vec{G}$ cont at $p \Rightarrow \vec{F} \pm \vec{G}$ & $\vec{F} \cdot \vec{G}$ cont at p.

Examples: 1) Absolute value Consider

$$|\cdot| : \mathbb{R}^k \longrightarrow \mathbb{R}$$
$$\vec{x} \longrightarrow |\vec{x}|.$$

Claim: $|\cdot|$ is cont on $\mathbb{R}^k$.

Pf: Let $\varepsilon > 0$, $\vec{p} \in \mathbb{R}^k$ Δ-ineq

$$|\vec{p} + \vec{h} - \vec{p}| \geq \big||\vec{p} + \vec{h}| - |\vec{p}|\big|.$$

$$\Rightarrow \big||\vec{p} + \vec{h}| - |\vec{p}|\big| \leq |\vec{h}|.$$

Choose $\delta = \varepsilon$.

$|\vec{h}| < \delta \Rightarrow ||\vec{p} + \vec{h}| - |\vec{p}|| < |\vec{h}| < \delta = \varepsilon$.

So $|\cdot|$ is cont. More generally, we have

Th^m 5.5: For any fixed $p \in X$, the function

$$f_p : X \longrightarrow \mathbb{R}$$
$$x \longrightarrow d(p, x).$$

is cont. on all of X.

2) Polynomials: The function $f : \mathbb{R} \to \mathbb{R}$.

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0.$$

where $a_k \in \mathbb{R}$ is cont.

Pf: By Th^m 4.4, enough to show $p_n(x) = x^n$ cont. at any $a \in \mathbb{R}$. Now

$$p_n(a) - p(a) = x^n - a^n = (x - a)(x^{n-1} + a x^{n-2} + \cdots + a^{n-1})$$

Let $\varepsilon > 0$. If $|x - a| < 1 \Rightarrow |x| < 1 + |a|$.

$$\Rightarrow |x^{n-1} + a x^{n-2} + \cdots + a^{n-1}| < (n-1)(1+|a|)^{n-1}.$$

$$\Rightarrow |p_n(a) - p(a)| < |x - a|(n-1)(1+|a|)^{n-1}.$$

If $\delta = \min\left(1, \dfrac{\varepsilon}{(n-1)(1+|a|)^{n-1}}\right)$

$$|x - a| < \delta \Longrightarrow |p_n(a) - p(a)| < \varepsilon.$$

$$\Rightarrow p_n(x) \text{ is cont at } x = a.$$

• <u>Creating New cont. function from old</u>

1) <u>Restrictions</u>:

<u>Th^m</u> 5.6 Let $f: X \to Y$ be cont. Then for any $E \subset X$.

$$f\big|_E : E \to Y \quad , \quad f\big|_E(x) = f(x) \ \forall \ x \in E.$$

is cont for the induced metric on E.

<u>Pf</u>: Let $p \in E$, and $\varepsilon > 0$. Since f cont on X, $\exists \ \delta$.
s.t $\forall \ x \in X$,

$$d_X(x, p) < \delta \implies d_Y(f(x), f(p)) < \varepsilon.$$

In particular $\forall \ x \in E$ s.t $d_X(x, p) < \delta$ we have

$$d_Y(f(x), f(p)) < \varepsilon.$$

But since $x, p \in E$, $d_X(x, p) = d_E(x, p)$. So $\forall \ x \in E$

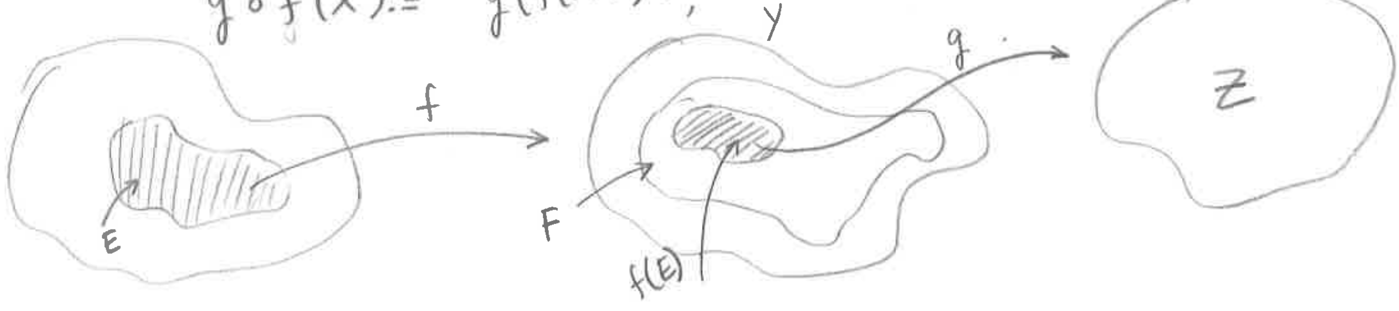
$$d_E(x, p) < \delta \implies d_Y(f\big|_E(x), f\big|_E(p)) = d_Y(f(x), f(p)) < \varepsilon.$$

So $f\big|_E$ is cont $\forall$ $p \in E$.

2) <u>Composition</u>: Let $E \subset X$, $F \subset Y$.

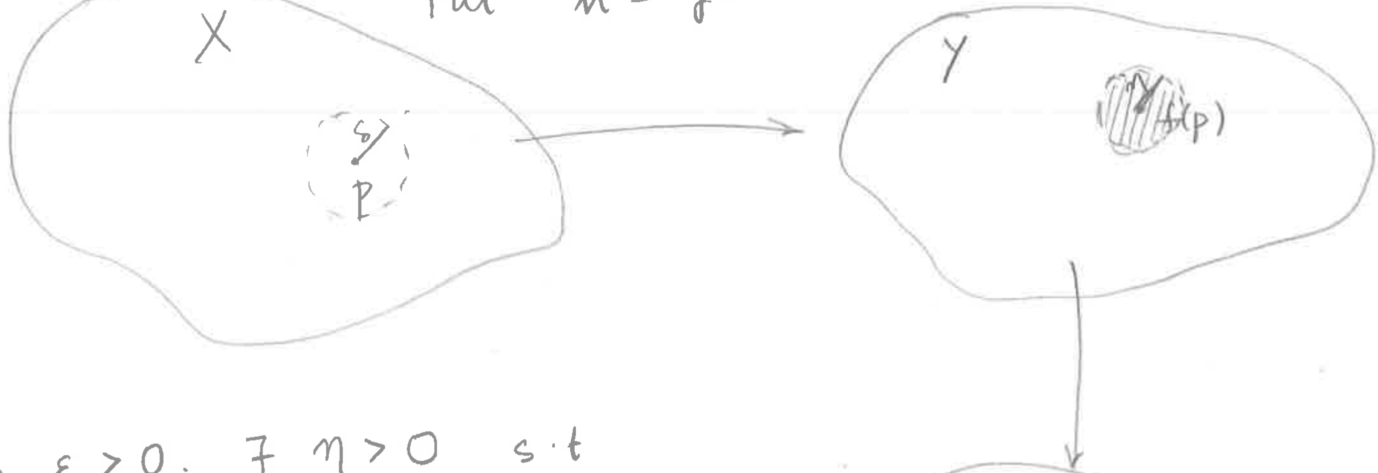
<u>Def^n</u>: If $f: E \to Y$, $g: F \to Z$ s.t Range$(f) \subset F$,
we define $g \circ f: E \to Z$ by

$$g \circ f(x) := g(f(x)),$$



$\underline{\text{Th}^m}$ 5.7 If f, g as above. Sps f is cont at $p \in E$ and g is cont at $f(p)$, then $g \circ f$ is cont. at p.

$\underline{\text{Pf}}$: Put $h = g \circ f$



Given $\varepsilon > 0$; $\exists\ \eta > 0$ s.t

(*) $d_Y(y, f(p)) < \eta \implies d_Z(g(y), g(f(p))) < \varepsilon$.

For this η, $\exists$ $\delta > 0$ s.t

(**) $d_X(x, p) < \delta \implies d_Y(f(x), f(p)) < \eta$.

Put $y = f(x)$ in (*) we obtain

$$d_X(x, p) < \delta \overset{**}{\implies} d_Y(f(x), f(p)) < \eta \overset{*}{\implies} d_Z(g(f(x)), g(f(p))) < \varepsilon$$

i.e

$$d_X(x, p) < \delta \implies d_Z(h(x), h(p)) < \varepsilon.$$

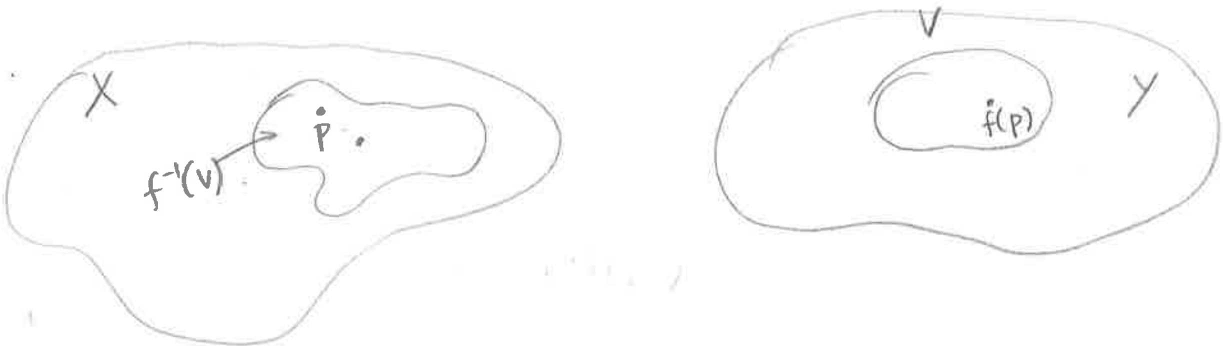
$\implies$ h is cont. at p

- Continuity and Open / Closed sets

Thm 5.8 1) A function $f: X \to Y$ is cont. if and only if $\forall$ open $V \subset Y$, $f^{-1}(V)$ is open in X.

2) A function $f: X \to Y$ is cont if and only if $\forall$ closed sets $E \subset Y$, $f^{-1}(E)$ is also closed in X.

Pf 1) $\Rightarrow$.



Let $V \subset Y$ open and $p \in f^{-1}(V)$. Then $f(p) \in V$.

V open $\Rightarrow$ $\exists$ $\varepsilon > 0$ s.t. $B_\varepsilon(f(p)) \subset V$.

Cont. $\Rightarrow$ $\exists$ $\delta > 0$ s.t $\forall$ $x \in B_\delta(p)$, $f(x) \in B_\varepsilon(f(p)) \subset V$.

$\Rightarrow B_\delta(p) \subseteq f^{-1}(V)$.

$\Rightarrow f^{-1}(V)$ is open.

$\Leftarrow$ Given $\varepsilon > 0$ and $p \in X$, $B_\varepsilon(f(p))$ is open in Y.

So $f^{-1}(B_\varepsilon(f(p)))$ is open in X.

Now $p \in f^{-1}(B_\varepsilon(f(p)))$. (since $f(p) \in B_\varepsilon(f(p))$).

So $\exists$ δ s.t $B_\delta(p) \subseteq f^{-1}(B_\varepsilon(f(p)))$.

i.e. if $x \in B_\delta(p)$ ($\Leftrightarrow d_X(x, p) < \delta$).

then $f(x) \in B_\varepsilon(f(p))$

that is

$$d_X(x,p) < \delta \implies d_Y(f(p), f(x)) < \varepsilon$$

$\implies f$ is cont. at $p \in X$.

2). Follows from 1) & the observation

$$f^{-1}(E^c) = [f^{-1}(E)]^c.$$

Rk: Image of an open set need not be open.
For eg. consider the constant function

$$f(x) = c$$

on $\mathbb{R}$.

- Continuity & Compactness:

Th^m 5.9 Sps $f: X \to Y$ is cont. Then for any compact set $K \subset X$, $f(K)$ is compact in Y.

Pf: Let $\{V_\alpha\}_{\alpha\in I}$ be a cover for $f(K)$.

$$f(K) \subseteq \bigcup_{\alpha \in I} V_\alpha$$

Then
$$K \subseteq \bigcup_{\alpha\in I} f^{-1}(V_\alpha).$$

So $\{f^{-1}(V_\alpha)\}$ is an open cover of K. Open since f is cont. and so $f^{-1}(V_\alpha)$ is open $\forall \alpha \in I$.

K compact $\implies \exists\ \alpha_1, \dots, \alpha_n$ s.t

$$K \subset \bigcup_{k=1}^{n} f^{-1}(V_{\alpha_k}).$$

$$\Rightarrow \quad f(K) \subset \bigcup_{k=1}^{n} f(f^{-1}(V_{\alpha_k}))$$

But $\quad f(f^{-1}(V_{\alpha_k})) \subset V_{\alpha_k}$ (note that V_{α_k} in general could be larger)

$$\Rightarrow \quad K \subset \bigcup_{k=1}^{n} V_{\alpha_k}.$$

So given open cover, we have extracted a finite sub-cover.

An important consequence is that a continuous real valued function attains its max and min.

<u>Def^n</u>: A function $\vec{f} : X \to \mathbb{R}^k$ is said to be <u>bounded</u> if $\exists$ $M > 0$ s.t

$$\|\vec{f}(x)\| \le M$$

$\forall$ $x \in X$.

<u>Th^m</u> 5.10 (Extremum value) Let X be compact.

1) If $f : X \to \mathbb{R}$ is <u>cont</u>, then f is bdd. Moreover if

$$M = \sup_{x \in X} f(x), \quad m = \inf_{x \in X} f(x).$$

Then $\exists$ $p, q \in X$ s.t $f(p) = M$, $f(q) = m$

2) If $f : X \to \mathbb{R}^k$ is cont., f is bdd. Moreover $\exists$ $p \in X$ s.t

$$\|f(p)\| = \sup_{x \in X} \|f(x)\|.$$

Pf: 1) $f(X)$ is compact and so is closed & bounded. So

$$M = \sup_{x \in X} f(x) \quad , \quad m = \inf_{x \in X} f(x)$$

exist. Moreover M, m will be limit points of $f(X)$. Since $f(X)$ is closed. $M, m \in f(X)$.
So $\exists$ $p, q \in X$ s.t $M = f(p)$, $m = f(q)$.

Note: We used the fact for any set $A \subset \mathbb{R}$, $\sup A$ & $\inf A$ are l.p of A.

2) Apply 1) to the composition

$$h(x) = |f(x)|.$$

Def^n: We call M, m above the maximum and minimum values of f on X. We call p and q the maxima and minima resp. (or simply max or min).

Example: $f: \mathbb{R} \to \mathbb{R}$.

$$f(x) = \frac{1}{1+x^2}.$$

Clearly f cont. & $0 < f \leq 1$. There is no minima of f on $\mathbb{R}$, since f can be made as close to 0 as we want, but $f(p) \neq 0$ $\forall$ $p \in \mathbb{R}$.

On the other hand

$$f\big|_{[0,1]} : [0,1] \to \mathbb{R}, \quad f(x) = \frac{1}{1+x^2}.$$

has a minima, namely $q = 1$.

Similarly it won't have a minima on $(0,1)$.

- <u>Uniform Continuity</u>

<u>Def^n</u> $f: X \to Y$ is said to be <u>uniformly</u> cont

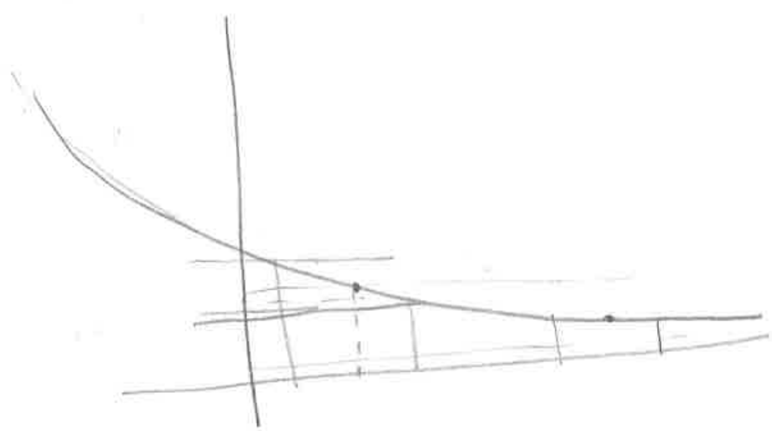
if $\forall\, \varepsilon > 0$, $\exists\, \delta = \delta(\varepsilon)$ s.t. $\forall\, p \in X$.

$$d_X(x, p) < \delta \implies d_Y(f(p), f(x)) < \varepsilon.$$

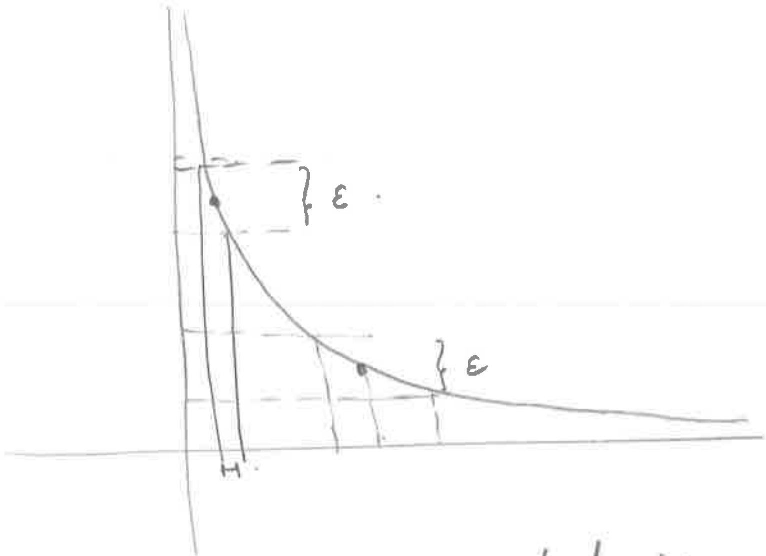
<u>Note</u> δ is independent of $p \in X$

<u>Example</u>: 1) Linear functions, $f(x) = ax + b$ are uniformly continuous

2) $f(x) = e^{-x}$ on $(0, \infty)$



3) $f(x) = 1/x$ on $(0, \infty)$



Here a much bigger δ is needed if x close to 0.

<u>Thm</u> 5.11: Sps. $f: X \to Y$ is cont. If X is compact, then f is <u>uniformly cont</u>.

<u>Pf</u>: Given $\varepsilon > 0$, f cont. $\Rightarrow \forall p \in X$, $\exists\ \delta_p$ s.t

(*) $\quad d_X(x, p) < \delta_p \Rightarrow d_Y(f(x), f(p)) < \varepsilon/2$.

Now $\quad X \subset \bigcup_{p \in X} B_{\delta_{p/2}}(p)$

X cmpt $\Rightarrow \exists\ p_1, \ldots, p_N$ s.t $X \subset \bigcup_{n=1}^{N} B_{\delta_{n/2}}(p_n)$.

Let $\quad \delta = \frac{1}{2}\min(\delta_{p_1}, \ldots, \delta_{p_N})$.

<u>Claim</u>: $\forall p \in X$, $d_X(x, p) < \delta \Rightarrow d_Y(f(x), f(p)) < \varepsilon$.

<u>Pf</u>: $p \in X \Rightarrow p \in B_{\delta_{p_n/2}}(p_n)$ for some p_n.

$\Rightarrow d_Y(f(p), f(p_n)) < \varepsilon/2$ by (*).

Moreover, Since $d_X(x,p) < \delta \Rightarrow d_X(x, p_1) < \delta_{p_n}/2$

$$\Rightarrow \quad d_X(x, p_n) < d(x,p) + d(p, p_n) < \delta_{p_n}/2 + \delta_{p_n}/2$$
$$= \delta_{p_n}$$

$$\overset{(*)}{\Rightarrow} \quad d_Y(f(x), f(p_n)) < \varepsilon/2$$

$$\Rightarrow \quad d_Y(f(p), f(x)) < d(f(p), f(p_n)) + d(f(p_n), f(x)) < \varepsilon$$

Done!

- <u>Continuity and Connectedness</u>

<u>Th^m</u> 5.12 If $f: X \to Y$ cont. If $E \subset X$ is connected, then $f(E)$ is connected subset of Y.

<u>Pf</u>: Sps not. Then we can find separated sets A, B s.t

1) $f(E) = A \cup B$
2) $A \cap \bar{B} = B \cap \bar{A} = \emptyset$, $A, B \neq \emptyset$

Let $G = E \cap f^{-1}(A)$, $H = E \cap f^{-1}(B)$.

Clearly $E = G \cup H$ & $G, H \neq \emptyset$

$A \subset \bar{A} \Rightarrow G \subset f^{-1}(\bar{A})$. $\bar{A}$ closed & f cont

$\Rightarrow \bar{G} \subset f^{-1}(\bar{A}) \Rightarrow f(\bar{G}) \subset \bar{A}$.

Also $f(H) = B$, $\bar{A} \cap B = \emptyset \Rightarrow \bar{G} \cap H = \emptyset$.

Similarly can show $G \cap \bar{H} = \emptyset$. So $E = G \cup H$ where G & H are separted $\Rightarrow$ E is disconnected

Contradiction!

Cor 5.13 (Intermediate value theorem) Let $f: [a,b] \to \mathbb{R}$ cont. Then for any c s.t $f(a) \leq c \leq f(b)$, $\exists$ $x \in [a,b]$ s.t

$$f(x) = c.$$

Pf: $f([a,b])$ is connected in $\mathbb{R}$. We showed earlier that $I \subseteq \mathbb{R}$ is connected in $\mathbb{R}$ if and only if it has the property that $p < c < q$, $p, q \in I$ $\Rightarrow$ $c \in I$. Done!

Example: Any cubic

$$p(x) = x^3 + ax + b = 0$$

has at least one real root. Since if $x \to -\infty$ then $p(x)$ becomes -ve. So it is -ve at some -ve x ($p(q_1)<0$). For instance if $a>1$, $b>1$, take $x = -a-b$. On the other hand as $x \to +\infty$ $p(x)$ becomes +ve. So there is a q_2 s.t $p(q_2) > 0$

i.e $$p(q_1) < 0 < p(q_2).$$

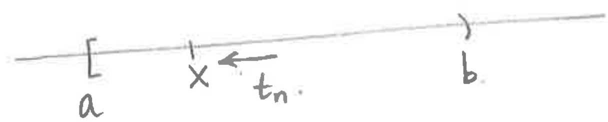
$$\Rightarrow \exists \ q \in (q_1, q_2) \text{ s.t } p(q) = 0.$$

• One Sided limits, Discontinuities

<u>Def^n</u>: Let $f : (a,b) \longrightarrow Y$ For $x \in [a, b)$. we write

$$f(x+) = q.$$

and say f has right side limit q if $f(t_n) \to q$ as $n \to \infty$ for any sequence $\{t_n\}$ in (x, b), $t_n \to x$



We can similarly define $f(x-)$.

Clearly. $\lim_{t \to x} f(x)$ exist $\iff$ $f(x+) = f(x-)$.

<u>Def^n</u>: Let $f : (a,b) \to Y$. We say f has dis cont. of <u>first kind</u> at $p \in (a,b)$ if $f(p+)$, $f(p-)$ exist but f is dis cont at p.

Else we say f has dis cont. of second kind.

<u>Rk</u>: f can have dis cont. of first kind at p if any of the foll 2 hold

1) $f(p+) \neq f(p-)$
2) $f(p+) = f(p-) \neq f(p)$.

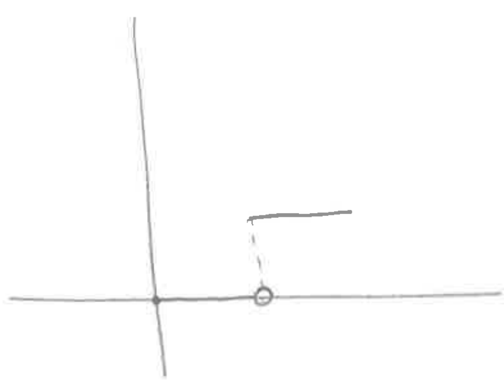
Examples 1). $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$

Neither $f(p+)$ nor $f(p-)$ exists for any p.
So discont. of second kind

2) $f(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$

f is cont. at 0 (Why?) but has discont. of second kind at every other point

3) $f(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & 1 \le x < 2 \end{cases}$



Clearly f is cont on $(0,1)$ & $(1,2)$.
$f(1+)$ & $f(1-)$ exist. In fact

$$f(1+) = 1, \quad f(1-) = 0.$$

So $f(1+) \neq f(1-)$ and f has discont of first kind at 1.

<u>Def</u>

Defn (Monotonic function) Let $f: (a,b) \to \mathbb{R}$.

1) f is called <u>increasing</u> if
$s < t \implies f(s) < f(t)$.

2) f is called <u>decreasing</u> if
$s < t \implies f(s) > f(t)$.

f is called monotonic if it is increasing or decreasing.

Thm 5.14 A monotonic function has <u>no</u> discont. of second type. That is, $\forall p \in (a,b)$, $f: (a,b) \to \mathbb{R}$, $f(p+)$ & $f(p-)$ always exist.

If f is increasing, then $f(p-) \leq f(p+)$

If f is decreasing, then $f(p-) \geq f(p+)$.

Pf: Sps f is increasing (the other case is similar)

For $p \in (a,b)$, let
$$A = \sup_{t \in (a,p)} f(t), \quad B = \inf_{t \in (p,b)} f(t).$$

Since f is increasing, $A \leq f(p) \leq B$.

Claim: $f(p-) = A$, $f(p+) = B$.

Given any $t_n \to p$, $\{t_n\}$ in (a,p). Clearly $f(t_n) < A$

Given $\varepsilon > 0$, $\exists$ δ s.t

$$A - \varepsilon < f(p - \delta) < A$$

Since $t_n \to p$, $\exists$ N s.t $\forall n > N$

$$t_n > p - \delta$$

Increasing $\Rightarrow$ $f(t_n) > f(p - \delta) > A - \varepsilon$

$\Rightarrow$ $A - \varepsilon \leq f(t_n) \leq A \quad \forall n > N$

$\Rightarrow$ $|f(t_n) - A| < \varepsilon \quad \forall n > N$

$\Rightarrow$ $f(t_n) \to A$ Done!

- Infinite limits & limits at infinity

Def^n For $f : (a, \infty) \to Y$, we say

$$\lim_{x \to \infty} f(x) = L$$

if $\forall \varepsilon > 0$, $\exists$ M s.t

$x > M \Rightarrow |f(x) - L| < \varepsilon$

Def^n: For $f : (a, b) \to \mathbb{R}$, we say

$$\lim_{t \to p} f(t) = \infty$$

if $\forall$ M, $\exists$ δ s.t

$|t - p| < \delta \Rightarrow f(t) > M$