# Sequences of functions

· <u>Main Problem</u>: Let $(X, d_X)$ & $(Y, d_Y)$ be metric spaces.

<u>Def$^n$</u>: Let $E \subset X$. We say that a seq$^n$ of functions $f_n : E \to Y$ <u>converges</u> (pointwise) to $f : E \to Y$ if

$$f(x) = \lim_{n\to\infty} f_n(x) \quad \forall\, x \in E,$$

and denote it by $f_n \to f$.

<u>Ques</u> 1) If $\{f_n\}$ are cont & $f_n \to f$, is $f$ cont?

2) Let $X, Y = \mathbb{R}$, $E = (a,b)$. If $f_n$ diff on $(a,b)$, is $f$ diff on $(a,b)$?

3) $X, Y \in \mathbb{R}$, $E = [a,b]$. If $f_n \in \mathcal{R}[a,b]$, is $f \in \mathcal{R}[a,b]$?

<u>Answer</u>: NO in each case.

<u>Rk</u>: 1) The first question can be rephrased in the foll way. If $x \to p$, is it true that $f(x) \to f(p)$ i.e

$$\lim_{x\to p} \lim_{n\to\infty} f_n(x) \overset{?}{=} \lim_{n\to\infty} f_n(p)$$

$$= \lim_{n\to\infty} \lim_{x\to p} f_n(x)$$

since $f_n$ is cont.

So the question is really about interchanging limits

2) If $f \in R[a,b]$ we would like to know if

$$\int_a^b f(t)\,dt \overset{?}{=} \lim_{n\to\infty} \int_a^b f_n(t)\,dt$$

$$\Leftrightarrow \quad \int_a^b \lim_{n\to\infty} f_n(t)\,dt \overset{?}{=} \lim_{n\to\infty} \int_a^b f_n(t)\,dt$$

i.e can integral & limits be interchanged.

Examples 1) $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = x^n$

Then $f_n$ is cont. on $[0,1]$ for each $n$

Also $f_n \to f$ where

$$f(x) = \begin{cases} 0 & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$$

which is discontinuous.

2) For $n = 1, 2, \ldots$, put

$$f_n(x) = \lim_{m\to\infty} (\cos n!\pi x)^{2m}$$

When $n!x$ is an integer, $f_n(x) = 1$ or else $f_n(x) = 0$.

Let us consider this on $[0,1]$

So clearly $f_n(x)$ has finite discontinuities, namely at rational $p/q$ where $q$ divides $n!$

$\Rightarrow \quad f_n(x) \in \mathcal{R}[0,1]$.

<u>Claim</u>: $f_n \to f$ where

$$f(x) = \begin{cases} 0, & x \in \mathbb{R} \setminus \mathbb{Q} \\ 1 & x \in \mathbb{Q} \end{cases}$$

<u>Pf</u>: If $x \in \mathbb{R}\setminus\mathbb{Q}$. Then $f_n(x) = 0 \;\forall\, n$

$\Rightarrow \quad \lim_{n\to\infty} f_n(x) = 0$.

If $x = p/q \in \mathbb{Q}$. Then for $n \geq q$,

$q$ divides $n!$ $\Rightarrow$ $f_n(x) = 1 \;\forall\, n \geq q$.

$\Rightarrow \quad f(x) = \lim_{n\to\infty} f_n(x) = 1$.

But $f \notin \mathcal{R}[0,1]$.

3) Let $f_n(x) = n^2 x (1-x^2)^n$. Then $f_n \to 0$ on $[0,1]$

A calculation shows

$$\int_0^1 f_n(x)\,dx = n^2 \int_0^1 x(1-x^2)^n\,dx \underset{x\,dx = -\frac{du}{2}}{\overset{1-x^2=u}{=}} \frac{n^2}{2}\int_0^1 u^n\,du$$

$$= \frac{n^2}{2n+2} \longrightarrow +\infty.$$

But $\int_0^1 f(x)\,dx = 0$.

So $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx \neq \int_0^1 \lim_{n\to\infty} f_n(x)\,dx$.

4). Consider $f_n(x) = \dfrac{\sin nx}{\sqrt{n}} \longrightarrow 0 =: f(x)$.

In particular $f'(0) = 0$.

But $f_n'(x) = \sqrt{n}\cos nx \Rightarrow f_n'(0) = \sqrt{n} \to +\infty$.

So $f_n \to f$, but $f'(0) \neq \lim_{n\to\infty} f_n'(0)$.

• <u>Uniform Convergence</u>

<u>Defn</u> We say that a seq of functions $f_n: E \to Y$ converges <u>uniformly</u> to $f$ on $E \subset X$ if $\forall\, \varepsilon > 0$, $\exists\, N = N(\varepsilon)$:

$n \geq N \Longrightarrow d_Y(f_n(x), f(x)) < \varepsilon$

$\forall\, x \in E$. We then denote $f_n \xrightarrow{u.c.} f$ on $E$.

<u>Rk</u>: 1) In <u>pointwise</u> <u>convergence</u>; the $N$ depends on $\varepsilon$ AND $x \in E$

2) Clearly uniform convergence $\Rightarrow$ pointwise conv. and the two limit functions are the same.

$\underline{\text{Def}^n}$: We say $\sum f_n(x)$ converges uniformly on $E$ if seq $\{S_n\}$ of partials sums converge uniformly.

$\underline{\text{Th}^m}$ (Cauchy criteria) Let $Y$ be complete. Then $f_n \xrightarrow{u.c} f$ on $E$ if & only if $\forall\ \varepsilon > 0$, $\exists\ N$ s.t

$$n, m \geq N \implies d_Y(f_n(x), f_m(x)) < \varepsilon \quad \forall\ x \in E \quad (*)$$

$\underline{\text{Pf}}$: $\underline{\implies}$ If $f_n \xrightarrow{u.c} f$ on $E$. Then given $\varepsilon > 0$, $\exists$ $N$ s.t.

$$n \geq N,\ x \in E \implies d_Y(f_n(x), f(x)) < \varepsilon/2.$$

But then if $n, m \geq N$, $x \in E$

$$d_Y(f_n(x), f_m(x)) \leq d_Y(f_n(x), f(x)) + d_Y(f_m(x), f(x))$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\Leftarrow$ For every $x$, $\{f_n(x)\}$ is a Cauchy seq.

$Y$ complete $\Longrightarrow$ $f_n(x) \longrightarrow f(x)$ for every $x \in E$.

<u>Claim</u> $f_n \to f$ uniformly.

Given $\varepsilon > 0$, $\exists$ $N$ s.t. $(*)$ holds. Fix $n$ & let $m \to \infty$ $\Rightarrow$ $\forall$ $x \in E$, $n \geq N$.

$$d_Y(f_n(x), f(x)) < \varepsilon.$$

Note, we are using the fact that $d_Y(\cdot, \cdot)$ is a cont. function

<u>Th$^m$</u>: Sps. $f_n \longrightarrow f$ on $E$. Put

$$M_n = \sup_{x \in E} d_Y(f_n(x), f(x)).$$

Then $f_n \xrightarrow{u.c} f$ on $E \iff \lim_{n\to\infty} M_n = 0$.

<u>Pf</u>: Immediate from def$^n$.

- <u>Uniform Continuity & Cont.</u> $(X, d_X)$ & $(Y, d_Y)$ be metric spaces.

<u>Th$^m$</u> Let $E \subset X$ and $f_n : E \to Y$ be a seq of functions on $E$. Then

$$\left.\begin{array}{l} \cdot\ f_n \to f \text{ u.c on } E \\ \cdot\ f_n \text{ cont. at } p \in E \end{array}\right\} \Longrightarrow f \text{ is cont at } p.$$

Pf: Let $x_m \to p$ in $E$.

Claim: $\lim_{m\to\infty} f(x_m) = f(p)$

Pf: Let $\varepsilon > 0$. $f_n \to f$ uniformly $\Rightarrow$ $\exists$ $N$ s.t

$\forall$ $x \in E$, $\forall$ $n \geq N$,

$$d_y(f_n(x), f(x)) < \frac{\varepsilon}{3}$$

In particular, for any $m$,

$$d_y(f_N(x_m), f(x_m)) < \frac{\varepsilon}{3}$$

$$d_y(f_N(p), f(p)) < \frac{\varepsilon}{3}.$$

Note that for any $m, n$

$$d_y(f(x_m), f(p)) < d_y(f(x_m), f_N(x_m)) + d_y(f_N(x_m), f_N(p)) + d_y(f_N(p), f(p)).$$

But since $f_N$ is cont, $\lim_{m\to\infty} f_N(x_m) = f_N(p)$.

So, $\exists$ $M$ s.t $\forall$ $m > M$,

$$d_y(f_N(x_m), f_N(p)) < \frac{\varepsilon}{3}.$$

$\Rightarrow$ if $m \geq M$,

$$d_y(f(x_m), f(p)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

This shows that

$$f(x_m) \to f(p) \quad \text{as} \quad m \to \infty$$

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<u>Cor</u>: $f_n : X \to \mathbb{R}$ s.t $\sum f_n$ converges uniformly

If $f_n$ cont. $\Longrightarrow$ $\sum_{n=1}^{\infty} f_n(x)$ is cont.

<u>Pf</u>: Let

$$S_n(x) = \sum_{k=1}^{n} f_k(x).$$

Since $f_n$ cont $\Longrightarrow$ $S_n$ cont, since <u>finite</u> sums preserve cont.

$\sum f_n$ conv uniformly $\overset{\Longrightarrow}{\text{Def}^n}$ $\{S_n\}$ conv. uniformly.

$$\overset{\text{Th}^m}{\Longrightarrow} \lim_{n\to\infty} S_n = \sum_{k=1}^{\infty} f_k(x) \quad \text{is cont.}$$

By essentially the <u>same</u> argument, one can prove the foll.

<u>Th$^m$</u>: Let $f_n : E \to Y$ s.t $f_n \to f$ <u>uniformly</u> on $E$. Let $p$ limit point of $E$, and suppose

$$A_n = \lim_{t\to p} f_n(t).$$

exists. Then $\{A_n\}$ is convergent &

$$\lim_{n\to\infty} A_n = \lim_{t\to p} f(p).$$

That is,

$$\lim_{n\to\infty} \lim_{t\to p} f_n(t) = \lim_{t\to p} \lim_{n\to\infty} f_n(t).$$

• Uniform Continuity and Integration

Th$^m$: Sps $f_n \in R[a,b]$ $\forall\ n = 1, 2, \ldots$ and sps $f_n \xrightarrow{u\cdot c} f$ on $[a,b]$. Then $f \in R[a,b]$ and

$$\int_a^b f(t)\,dt = \lim_{n\to\infty} \int_a^b f_n(t)\,dt.$$

Pf: Put

$$\varepsilon_n = \sup_{t\in[a,b]} |f(t) - f_n(t)|$$

Then $f_n \xrightarrow{u\cdot c} f \iff \varepsilon_n \to 0$ as $n \to \infty$.

By definition of $\varepsilon_n$, for all $n$, $\forall\ t \in [a,b]$

$$f_n(t) - \varepsilon_n < f(t) < f_n(t) + \varepsilon_n$$

$$\Rightarrow \underline{\int_a^b}(f_n - \varepsilon_n)\,dt < \underline{\int_a^b} f(t)\,dt < \overline{\int_a^b} f(t)\,dt < \overline{\int_a^b}(f_n + \varepsilon_n)\,dt$$

$$f_n \in R[a,b] \Rightarrow \underline{\int_a^b} f_n = \overline{\int_a^b} f_n = \int_a^b f_n$$

$$\Rightarrow \int_a^b f_n(t)\,dt - \varepsilon_n(b-a) < \underline{\int_a^b} f(t)\,dt < \overline{\int_a^b} f(t)\,dt < \int_a^b f_n(t)\,dt + \varepsilon_n(b-a)$$

$(*)$.

$$\Rightarrow \quad 0 \leq \overline{\int_a^b} f(t)\,dt - \underline{\int_a^b} f(t)\,dt < 2\varepsilon_n(b-a).$$

Let $\varepsilon > 0$, and $N$ s.t $\varepsilon_N < \varepsilon/2(b-a)$

$$\Rightarrow \quad \overline{\int_a^b} f(t)\,dt - \underline{\int_a^b} f(t)\,dt < \varepsilon.$$

True $\forall\ \varepsilon > 0 \Rightarrow \overline{\int_a^b} f(t)\,dt = \underline{\int_a^b} f(t)\,dt$

$$\Rightarrow \quad f \in R[a,b].$$

Now from (*)

$$\int_a^b f_n(t)\,dt - \varepsilon_n(b-a) \leq \int_a^b f(t)\,dt \leq \int_a^b f_n(t)\,dt + \varepsilon_n(b-a)$$

Letting $n \to \infty$, $\varepsilon_n \to 0$

$$\Rightarrow \quad \int_a^b f(t)\,dt = \lim_{n\to\infty} \int_a^b f_n(t)\,dt.$$

<u>Cor</u>: If $\sum f_n$ converges uniformly & $f_n \in R[a,b]$ $\forall n$. Then $\sum f_n \in R[a,b]$ &

$$\int_a^b \sum_{n=1}^{\infty} f_n(t)\,dt = \sum_{n=1}^{\infty} \int_a^b f_n(t)\,dt.$$

In particular $\sum_{n=1}^{\infty} \int_a^b f_n(t)\,dt$ converges.

## • Uniform Convergence and Differentiation

**Example**: Let $f_n(t) = \sqrt{t^2 + 1/n}$ on $[-1, 1]$.

**Claim**: $f_n(t) \longrightarrow |t|$ uniformly on $[-1, 1]$

**Pf**: Let $f(t) = |t|$.

$$|f_n(t) - f(t)| = \sqrt{t^2 + 1/n} - |t| = \left(\sqrt{t^2 + 1/n} - |t|\right) \cdot \frac{\sqrt{t^2+1/n} + |t|}{\sqrt{t^2+1/n} + |t|}$$

$$= \frac{t^2 + 1/n - |t|^2}{\sqrt{t^2+1/n} + |t|}$$

But $\sqrt{t^2+1/n} + |t| > 1/\sqrt{n}$.

$$\Rightarrow \quad |f_n(t) - f(t)| < \frac{1/n}{1/\sqrt{n}} < \frac{1}{\sqrt{n}}.$$

Given $\varepsilon > 0$, choose $N > 1/\varepsilon^2$ (i.e. $1/\sqrt{N} < \varepsilon$). Then

$n \geq N \Rightarrow \forall \; t \in [-1, 1]$,

$$|f_n(t) - f(t)| < \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{N}} < \varepsilon.$$

Also, $f_n$ is diff on $[-1, 1]$ for each $n$ <u>BUT</u>

$f$ is not diff. at $t = 0$.

So a direct analog of the theorems on cont. & integrability is not valid.

$\underline{\text{Th}^m}$: Sps $f_n$ is a seq$^n$ of diff functions on $(a, b)$. $\boxed{a, b \in (-\infty, \infty)}$

s.t $\{f_n(x_0)\}$ converges for some $x_0 \in (a, b)$

If $\{f_n'\}$ converges <u>uniformly</u> on $(a, b)$. Then

1) $\{f_n\}$ converges uniformly on $(a, b)$ to a limit $f$.

2) $f$ is diff on $(a, b)$ &

$$f'(x) = \lim_{n\to\infty} f_n'(x) \quad \forall \quad x \in (a, b).$$

<u>Pf</u>: 1). Given $\varepsilon > 0$, $\exists$ $N$ s.t

$$|f_n(x_0) - f_m(x_0)| < \frac{\varepsilon}{2} \quad \forall \quad n, m \geq N.$$

<u>AND</u>

$$|f_n'(s) - f_m'(s)| < \frac{\varepsilon}{2(b-a)} \quad \forall \; s \in (a, b).$$

By MVT applied to $f_n - f_m$, for any $x, t \in (a, b)$

$\exists$ $s$ between $x$ and $t$ s.t

$$|f_n(x) - f_m(x) - f_n(t) + f_m(t)| \leq |f_n'(s) - f_m'(s)|\,|b-a|.$$

$$< \frac{\varepsilon\,|x-t|}{2(b-a)} = \frac{\varepsilon}{2} \; (*)$$

Then applying the ineq to $t = x_0$ & $\Delta$-ineq

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f_m(x) - f_n(x_0) + f_m(x_0)|$$

$$+ |f_n(x_0) - f_m(x_0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So $\{f_n\}$ is uniformly Cauchy and converges uniformly on $(a,b)$.

Let $$f(x) = \lim_{n\to\infty} f_n(x).$$

2). Fix $x \in (a,b)$, and define

$$\varphi_n(t) = \frac{f_n(t) - f_n(x)}{t - x}, \quad \varphi(t) = \frac{f(t) - f(x)}{t - x}.$$

Then for $n = 1, 2, \dots$, $\lim_{t\to x} \varphi_n(t) = f'(x)$.

By $(*)$.

$$|\varphi_n(t) - \varphi_m(t)| = \frac{|f_n(t) - f_m(t) - f_n(x) + f_m(x)|}{|t - x|}$$

$$\leq \frac{\varepsilon}{2(b-a)}. \quad \forall\ n, m \geq N.$$

$\Longrightarrow$ $\{\varphi_n(t)\}$ conv. uniformly on $t \neq x$.

But $f_n \xrightarrow{u.c} f \Longrightarrow \varphi_n \xrightarrow{u.c} \varphi$ on $(a,b)\setminus\{x\}$.

Applying theorem on interchanging limits

$$f'(x) = \lim_{t\to x} \varphi(t) = \lim_{t\to x}\lim_{n\to\infty} \varphi_n(t) = \lim_{n\to\infty}\lim_{t\to x} \varphi_n(t)$$

$$= \lim_{n\to\infty} f_n'(x).$$

Since this last limit exist, $\lim_{t\to x} \varphi(t)$ exists and hence $f$ is diff at $x \in (a,b)$ &

$$f'(x) = \lim_{n\to\infty} f_n'(x).$$