# MATH 104: Lecture 4

- **Cardinality of $\mathbb{Q}$ and $\mathbb{R}$.**

**Th$^m$**: Let $\{E_n\}_{n=1}^{\infty}$ be a seq$^n$ of countable sets. Then

$$S = \bigcup_{n=1}^{\infty} E_n$$

is also countable.

**Pf**: Let $E_n = \{x_{nk}\}_{k=1}^{\infty}$, and consider

$$\begin{array}{cccc} x_{11} & x_{12} & x_{13} & \cdots \\ x_{21} & x_{22} & x_{23} & \cdots \\ x_{31} & x_{32} & x_{33} & \cdots \end{array}$$

Arrange them as a seq. (thinking of each entry as distinct)

$$T = \{x_{11}, x_{21}, x_{12}, x_{31}, x_{22}, x_{13}, \cdots\}$$

Then $T$ is countable. Clearly $S \subset T$. Also
$E_1 \subset S \Rightarrow S$ is infinite
$\Rightarrow S$ is also countable.

**Cor**: $\mathbb{Q}$ is countable.

**Pf**: Let $E_n = \{p/n \mid p \in \mathbb{Z}\}$.
Then $E_n \sim \mathbb{Z}$ and hence countable.

$$\Rightarrow \mathbb{Q} = \bigcup_{n=1}^{\infty} E_n \text{ is countable}$$

$\underline{Th^m}$: The set $S$ of all numbers in $(0,1)$ is uncountable

$\underline{Pf}$: We proceed by contradiction. Sps. $(0,1)$ is countable. Then we can write

$$(0,1) = \{x_1, x_2, \dots\}.$$

We write $x_n$ as a decimal

$$x_n = 0 \cdot d_{n1} d_{n2} d_{n3} \dots = \sum_{i=1}^{\infty} \frac{d_{ni}}{10^i}$$

We use the convention that if the decimal expansion terminates we append all zeroes. So for instance we write

$$\frac{1}{2} = 0 \cdot 5000 \dots$$

and $\underline{\text{NOT}}$ $0 \cdot 49999 \dots$

Then consider the number

$$y = 0 \cdot a_1 a_2 \dots$$

s.t

$$a_i = \begin{cases} 1 & \text{if } d_{ii} \neq 1 \\ 2 & \text{if } d_{ii} = 1 \end{cases}$$

Clearly $y \in (0,1)$.
Then $y$ differs from $x_n$ in the $n^{th}$ decimal place $\Rightarrow y \notin (0,1)$ since we assumed that

the list $\{x_1, \ldots\}$ is exhaustive.

This is a contradiction.

$\Longrightarrow$ $(0,1)$ is uncountable.

<u>Rk</u>: This is the famous Cantor diagonalization argument.

<u>Cor</u>: $\mathbb{R}$ is uncountable.

<u>Pf</u>: If it is countable, then $(0,1)$ be infinite & $(0,1) \subset \mathbb{R}$ would also have to be countable

Contradiction!

## <u>Ch 2</u>: Metric Spaces.

<u>Def$^n$</u>: A set $X$ is said to be a <u>metric space</u>. if there is a function

$$d: X \times X \longrightarrow \mathbb{R}_+$$
$$(p, q) \longrightarrow d(p, q).$$

called the <u>distance function</u> s.t

(1) (positive definite) $\forall\ p, q \in X$,

$$d(p, q) \geq 0$$

and $d(p,q) = 0$ if and only if $p = q$.

(2) (Symmetry) $d(p,q) = d(q,p)$.

(3) (Triangle ineq) $\forall\ p, q, r \in X$.

$$d(p, q) \leq d(p, r) + d(r, q).$$

**Examples** 1) $(\mathbb{R}, |\cdot|)$ where we define

$$d(s,t) = |s-t| = |t-s| \geq 0.$$

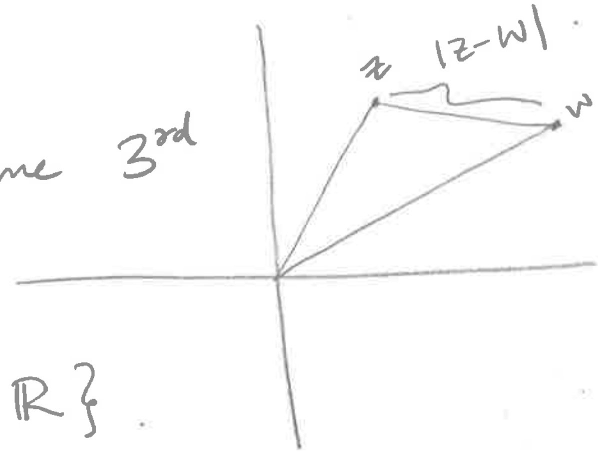

Then $\quad |s-t| \leq |s-u| + |t-u|$.

eq if $u$ lies between $s$ & $t$.

Clearly $|s-t| = 0 \Leftrightarrow s = t$.

2) $(\mathbb{C}, |\cdot|)$. For $z, w \in \mathbb{C}$, define

$$d(z,w) = |z-w|$$




To check $\Delta$-ineq can assume $3^{rd}$ point is $0$.

3) $\mathbb{R}^k = \{\vec{x} = (x_1, \dots, x_k) \mid x_i \in \mathbb{R}\}$.

Define $|\vec{x}| = \sqrt{x_1^2 + x_2^2 + \dots + x_k^2}$,

and $\quad d(\vec{x}, \vec{y}) = |\vec{x} - \vec{y}|$.

**Exercise**: $\quad d(\vec{x}, \vec{y}) \leq d(\vec{x}, \vec{0}) + d(\vec{0}, \vec{y})$.

4) <u>(Pathological example)</u>. $X$ be any set.

Define
$$d(x,y) = \begin{cases} 0 & \text{if } x = y. \\ 1 & \text{if } x \neq y. \end{cases}$$

<u>Exercise</u>: Check that this is a metric space.

# MATH 104: Lecture 5

- Examples (cont.):

5) $(\mathbb{Q}, |\cdot|)$ We can again define a distance

$$d(p,q) = |p-q|.$$

Note that $\mathbb{Q} \subset \mathbb{R}$ and the distance function is the same. We say $(\mathbb{Q}, |\cdot|)$ is a metric subspace of $(\mathbb{R}, |\cdot|)$

Def$^n$: Let $(X, d)$ be a metric space. For any set $A \subset X$, we can define a metric

$$d_A(x, y) = d(x, y).$$

We then say that $(A, d_A)$ is a subspace of $(X, d)$.

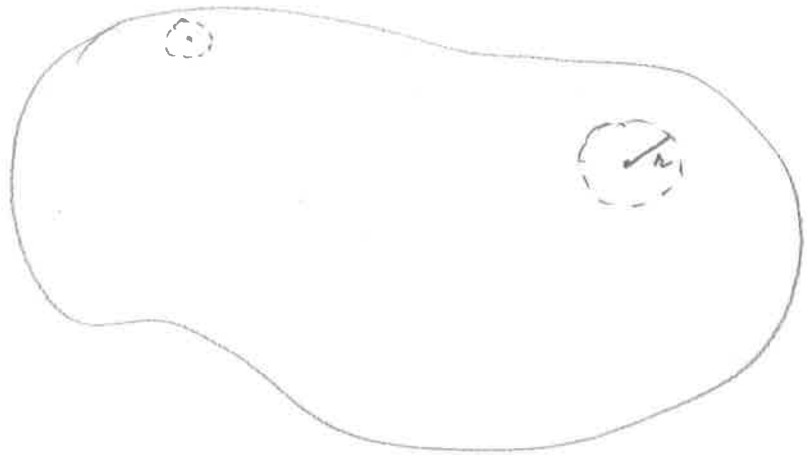
- Open and closed sets    Let $(X, d)$ be a metric sp.

Def$^n$: For $p \in X$, $r > 0$, the (open) ball of radius '$r$' around $p$ is the set

$$B_r(p) = \{x \in X \mid d(p, x) < r\}.$$

Def$^n$: 1) A set $U \subset X$ is called open if $\forall\, p \in U$, $\exists\, r$ (depending on $p$) s.t.

$$B_r(p) \subset U.$$



2) A set $A \subset X$ is <u>closed</u> if

$$A^c = \{x \in X \mid x \notin A\}.$$

is open.

<u>Rk</u>: By convention we let the empty set $\phi$ be both open & closed. $\Rightarrow$ $X$ is both open & closed.

<u>Examples</u> 1) $(a,b) \subset \mathbb{R}$ is open; $[a,b] \subset \mathbb{R}$ is closed.

2) $[\sqrt{2}, \sqrt{3}]$ is closed in $\mathbb{R}$, but $[\sqrt{2}, \sqrt{3}] \cap \mathbb{Q}$ is open in $\mathbb{Q}$, since

$$\{p \in \mathbb{Q} \mid \sqrt{2} \leq p \leq \sqrt{3}\} = \{p \in \mathbb{Q} \mid \sqrt{2} < p < \sqrt{3}\}.$$

<u>Def$^n$</u> $A \subset X$ ~~is dense if $\forall p \in A$ $\forall r > 0$ $B_r(p)$ contains points in $A^c$~~

**Prop.** (a) The open ball $B(p,r)$ is an open set.

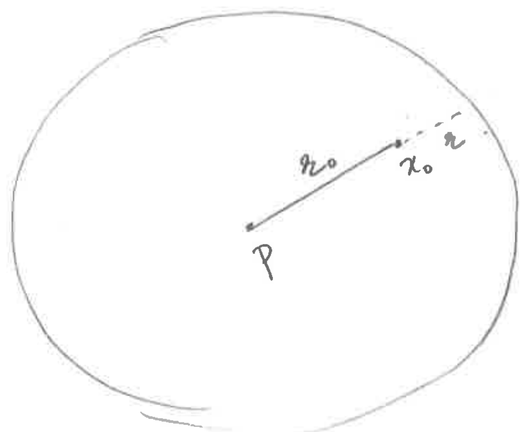
(b) The closed ball

$$\overline{B(p,r)} = \{x \in X \mid d(p,x) \le r\}$$

is closed.

**Proof:** (a) Let $x_0 \in B(p,r)$ i.e. $r_0 := d(p,x_0) < r$.

Let $\delta = r - r_0$.




**Claim:** $B(x_0, \delta) \subset B(p,r)$.

**Pf:** Let $x \in B(x_0, \delta) \Rightarrow d(x, x_0) < \delta = r - r_0$.

$$\Rightarrow \quad d(p,x) \le d(p,x_0) + d(x_0,x)$$
$$< r_0 + (r - r_0) = r$$

$\Rightarrow x \in B(p,r)$

So claim is proved, and $B(p,r)$ is open.

(b) $\overline{B(p,r)}^c = \{x \in X \mid d(p,x) > r\}$.

Sps $x_0 \in \overline{B(p,r)}^c$ s.t $r_0 = d(p,x_0) > r$

**Claim** If $\delta = r_0 - r$; $B(x_0, \delta) \subset \overline{B(p,r)}^c$.

To see this,

$$d(p, x) \geq d(p, x_0) - d(x_0, x)$$
$$= r_0 - d(x_0, x)$$

$d(x_0, x) \leq r_0 - r \Rightarrow -d(x_0, x) > r - r_0$

$\Rightarrow \quad d(p, x) > r_0 + r - r_0 = r.$

$\Rightarrow \quad d(p, x) > r \Rightarrow x \in \overline{B(p, r)}^c.$

$\Rightarrow \quad \overline{B(p, r)}^c$ is open $\Rightarrow \overline{B(p, r)}$ is closed.

<u>Th$^m$</u> 1) If $\{G_\alpha\}_{\alpha \in I}$ is an (arbitrary) collection of open sets in $X$.

$$V = \bigcup_{\alpha \in I} G_\alpha$$

is also open.

2). If $\{G_k\}_{k=1}^n$ is a finite collection of open sets in $X$.

$$W = \bigcap_{k=1}^n G_k$$

is open.

<u>Pf</u>: 1). Let $p \in V \Rightarrow p \in G_\alpha$ for some $\alpha \in I$.

$G_\alpha$ open $\Rightarrow \exists\ r$ s.t $B(p, r) \subset G_\alpha$

$\Rightarrow B(p, r) \subset V$. Done!

2). Let $p \in W \Rightarrow p \in G_k$ for $k = 1, 2, \dots, n$.

For each $k$, $G_k$ is open $\Rightarrow \exists\ r_k > 0$ s.t

$$B(p, r_k) \subset G_k.$$

Let $r = \min\{r_1, r_2, \dots, r_n\}$.

Then $B(p,r) \subset B(p, r_k) \subset G_k \quad \forall k$.

$\Rightarrow \quad B(p,r) \subset W \Rightarrow W$ is open.

We have an analogous result for closed sets

Cor 1) If $\{F_\alpha\}_{\alpha \in I}$ is an arbitrary collection of closed sets in $X$ then

$$A = \bigcap_{\alpha \in I} F_\alpha$$

is also closed.

2) If $\{F_k\}_{k=1}^{n}$ is a finite collection of closed sets in $X$, then

$$B = \bigcup_{k=1}^{n} F_k$$

is also closed.

Pf: Follows from the set theory observation that

$$\left(\bigcap F_\alpha\right)^c = \bigcup F_\alpha^c$$

$$\left(\bigcup F_\alpha\right)^c = \bigcap F_\alpha^c$$

and applying theorem.

Example: Thm not true for arbitrary intersections

For instance $U_n = (-1/n, 1/n)$ is open $\forall n$

$$\bigcap_{n=1}^{\infty} U_n = \{0\}$$ which is NOT open.

## • Interior points & limit points

**Def$^n$**: Let $E \subset X$

1) $p \in E$ is called an <u>interior point</u> of $E$ if $\exists\ r > 0$ s.t $B(p, r) \subset E$.
2) The set of interior points is called the interior & denoted by $E^\circ$.
3) $p \in E$ is called a <u>limit point</u> if $\forall\ r > 0$, $B(p, r) \cap E$ has at least one point other than $p$.
4) The <u>closure</u> $\overline{E}$ is defined to be the set of all points in $E$ along with all limit points.
5) The boundary $\partial E$ is defined to be

$$\partial E = \overline{E} \setminus E^\circ = \{ x \in \overline{E} \mid x \notin E^\circ \}.$$

**Examples** 1) $E = [0, 1)$.

$E^\circ = (0, 1)$, $\overline{E} = [0, 1]$

$\partial E = \{0, 1\}$.

2) $E = \{ 1/n \mid n = 1, 2, \dots \}$. $E^\circ = \phi$ $\overline{E} = E \cup \{0\}$

Only l.p is $0$

# MATH 104 - Lecture 6

• <u>Rk</u>: $E$ is open $\iff$ $E = E^\circ$.

<u>Th$^m$</u> $E$ is closed $\iff$ $E = \overline{E}$.

<u>Pf</u>: $\Rightarrow$. Sps $E$ is closed and let $p \in E^c$. Then $E^c$ open and so, $\exists\ r_p$ s.t $B_{r_p}(p) \subset E^c \Rightarrow p$ cannot be a limit point $\Rightarrow$ all limit points of $E$ are contained in $E \Rightarrow E = \overline{E}$.

$\Leftarrow$. Sps $E = \overline{E}$. We then show $E^c$ is open.

Let $p \in E^c$. $\Rightarrow p$ is not a limit point of $E$

$\Rightarrow \exists\ r_p$ s.t $B_{r_p}(p) \cap E = \phi \Rightarrow B_{r_p}(p) \subset E^c$

This is true for all $p \in E^c \Rightarrow E^c$ is open

$\Rightarrow E$ is closed.

• <u>Dense Sets</u>

<u>Def$^n$</u>: $E \subset X$ is called <u>dense</u> if $\overline{E} = X$.

<u>Th$^m$</u>: $\mathbb{Q}$ is dense in $\mathbb{R}$

<u>Pf</u>: We need to show that any $x \in \mathbb{R}$ is either in $\mathbb{Q}$ or a limit point. Infact we claim

<u>Claim</u>: Any $x \in \mathbb{R}$ is a limit point of $\mathbb{Q}$.

<u>Pf</u>: Let $x \in \mathbb{R}$ and $r > 0$. Then $x + r > x$.

$\Rightarrow \exists\ p \in \mathbb{Q}$ s.t $x < p < x + \varepsilon$.

$\Rightarrow p \in B_\varepsilon(x) \cap \mathbb{Q}$.

$\Rightarrow \forall\ \varepsilon > 0$, $B_\varepsilon(x) \cap \mathbb{Q}$ contains a pt other than $x$. So, $x$ is a limit point

- Compact Sets: $(X, d)$ metric space

Def$^n$: An open cover for $E \subseteq X$ is a collection of open sets $\{G_\alpha\}_{\alpha \in I}$ s.t

$$E \subset \bigcup_{\alpha \in I} G_\alpha.$$

Def$^n$: A set $K \subset X$ is called compact if every open cover has a finite sub-cover. That is, for any open cover $\{G_\alpha\}$, $\exists\ \alpha_1, \dots, \alpha_n$ s.t

$$K \subset \bigcup_{k=1}^{n} G_{\alpha_k}.$$

Example: $K = \{1/n \mid n = 1, 2, \dots\} \cup \{0\}$ is compact.

Pf: Let $\{G_\alpha\}_{\alpha \in I}$ be an open cover. Let $\alpha_0$ s.t $0 \in G_{\alpha_0}$. Since $G_{\alpha_0}$ is open, $\exists\ \varepsilon$ s.t $B_\varepsilon(0) \subseteq G_{\alpha_0}$.

$\Rightarrow \exists\ N > 0$ (take $N > 1/\varepsilon$) s.t $1/n \in G_{\alpha_0}\ \forall\ n > N$.

Let $\alpha_1, \dots, \alpha_N$ s.t $1/k \in G_{\alpha_k}$, $k = 1, 2, \dots, N$

Then $$K \subset \bigcup_{k=0}^{N} G_{\alpha_k}$$

So we have extracted a finite sub-cover.

$\underline{\text{Th}}^m$ (1) Compact subsets are closed

(2) Closed subsets of compact sets are compact.

$\underline{\text{Pf}}$: (1) Let $K \subset X$ be compact. We show $K^c$ is open.

Let $p \in K^c$.

For each $q \in K$, let $r_q = d(p,q)/2$.

Then $B_{r_q}(p) \cap B_{r_q}(q) = \phi$.

Clearly $K \subset \bigcup\limits_{q \in K} B_{r_q}(q)$. Compactness $\Rightarrow \exists\ q_1, \dots, q_n \in K$

s.t $\quad K \subseteq \bigcup\limits_{k=1}^{n} B_{r_{q_k}}(q_k) := W$.

Let $\quad r_p = \min(r_{q_1}, \dots, r_{q_n})$.

$\underline{\text{Claim}}$: $B_{r_p}(p) \cap W = \phi$. In particular $B_{r_p}(p) \subset K^c$.

$\underline{\text{Pf}}$: Let $x \in W$. Then $d(x, q_k) < r_{q_k}$ for some $k = 1, \dots, n$

$\Delta$-ineq $\Rightarrow \quad d(p, q_k) \leq d(p,x) + d(x, q_k)$.

$$\Rightarrow \quad d(p,x) \geq d(p,q_k) - d(x,q_k).$$

$$= 2r_{q_k} - d(x,q_k) > 2r_{q_k} - r_{q_k}$$

$$= r_{q_k}$$

$$\Rightarrow \quad d(p,x) > r_{q_k} \geq r_p.$$

$\Rightarrow \quad x \notin B_{r_p}(p)$. So intersection is empty.

This shows $K^c$ is open & hence $K$ is closed.

(2). Let $K$ be compact & $E \subset K$ closed.

Sps $\{G_\alpha\}$ is an open cover of $E$.

Then $\{\{G_\alpha\}, E^c\}$ is an open cover of $K$ (since $E^c$ is open).

$\overset{\text{compactness}}{\Longrightarrow}$ $\exists$ $\alpha_1, \dots, \alpha_n$ s.t

$$K \subset \bigcup_{k=1}^{n} G_{\alpha_k} \cup E^c.$$

$$\Rightarrow \quad E \subset \bigcup_{k=1}^{n} G_{\alpha_k} \quad \text{since } E \cap E^c = \emptyset \ \& \ E \subset K.$$

So every open cover has a finite sub-cover.

**Thm**: If $\{K_\alpha\}_{\alpha \in I}$ is a collection of compact subsets of $X$ s.t any finite intersection is non-empty. Then

$$\bigcap_{\alpha \in I} K_\alpha$$

is non-empty.

**Pf**: Sps $\bigcap_{\alpha \in I} K_\alpha$ is empty. Fix $K_1 = K_{\alpha_1}$. Then $\forall x \in K_1$, there is some $\alpha(x)$ s.t $x \notin K_{\alpha(x)}$. Let $G_\alpha = K_\alpha^c$.

Then $\{G_\alpha\}$ is an open cover of $K$.

Compactness $\Rightarrow$ $\exists$ $\alpha_1, \dots, \alpha_n$ s.t $K_1 \subset \bigcup_{k=1}^{n} G_{\alpha_k}$

$$\Rightarrow \quad K^c \supseteq \left(\bigcup_{k=1}^{n} G_{\alpha_k}\right)^c = \bigcap_{k=1}^{n} G_{\alpha_k}^c = \bigcap_{k=1}^{n} K_{\alpha_k}.$$

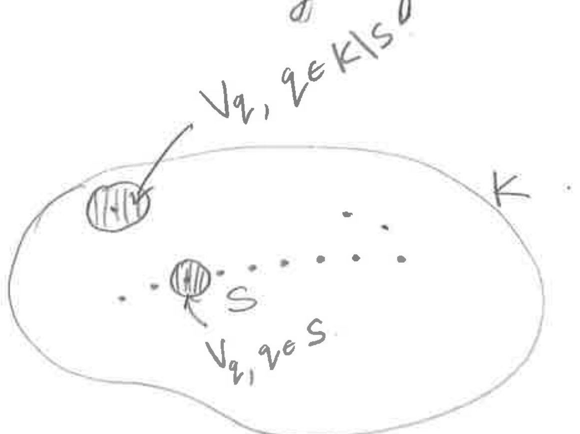
$$\Rightarrow \quad K_1 \cap K_{\alpha_1} \cap \dots \cap K_{\alpha_n} = \emptyset$$

contradicting the hypothesis.

· <u>Limit point compactness</u>.

<u>Thm</u>: Let $K \subset X$ compact. Then any infinite subset $S \subset K$ has a limit point in $K$.

<u>Pf</u>: Sps $S$ has no limit point in $K$.
In particular, each point in $S$ is isolated.
Form a covering of $K$ in the foll way.




Let $V_q$ be a ball around $q \in K$ s.t

(1). If $q \notin S$, $V_q \cap S = \phi$.

(2) If $q \in S$, $V_q \cap S = \{q\}$.

Then $K = \bigcup_{q \in K} V_q$.

If there is a finite sub-cover, then $S$ will have to be finite. So $K$ is not compact & we get a contradiction

**Def$^n$**: We say $K \subset X$ is limit point $\overset{(l\cdot p)}{\text{compact}}$ if every infinite subset of $K$ has a $l \cdot p$ in $K$.

**Rk**: In fact $K$ is compact $\Longleftrightarrow$ $K$ is $l \cdot p$ compact

$\xRightarrow{\text{Th}^m}$

$\xLeftarrow{\text{Assignment}}$

## • Compact Sets in $\mathbb{R}^k$

**Th$^m$** The <u>k-cell</u>

$$I = \{ \vec{x} \in \mathbb{R}^k \mid a_i \le x_i \le b_i \}.$$

is compact in $\mathbb{R}^k$.

**Lemma**: If $\{I_n\}$ is a sequence of $k$-cells s.t

$$I_{n+1} \subseteq I_n$$

then $\bigcap_{n=1}^{\infty} I_n$ is non-empty.

**Pf**: <u>Step 1</u> $k = 1$. Then $I_n = [a_n, b_n]$.

The set $\{a_n\}$ is bounded since $a_n \le b_1 \ \forall n$.

$\Rightarrow$ sup exist. So let

$$t^* = \sup a_n.$$

<u>Claim</u>: $t^* \in \bigcap_{n=1}^{\infty} I_n$.

Clearly $a_n \le t^* \quad \forall n$.

Also $a_m \le b_n \quad \forall \ n, m$.

$\Rightarrow \quad t^* \le b_n \ \forall \ n$.

$\Rightarrow \quad t^* \in [a_n, b_n] = I_n \ \forall \ n$.

<u>Step 2</u> $k > 1$. Let $I_n = \{\vec{x} \mid a_{n,i} \le x_i \le b_{n,i}\}$.

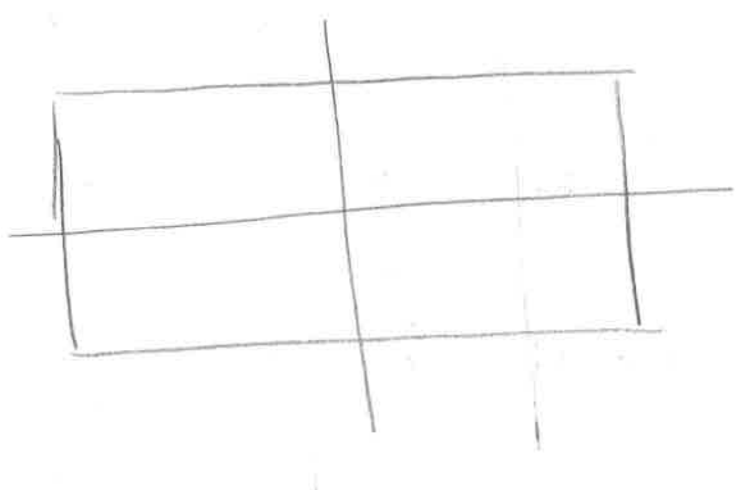
For a fixed component '$i$', we let

$$I_n^{(i)} = [a_{n,i}, b_{n,i}]$$

Then $\exists \ t_i^* \in [a_{n,i}, b_{n,i}] \ \forall \ n$

$\Rightarrow \quad \vec{t} = (t_1^*, \dots, t_k^*) \in I_n \ \forall \ n$.

<u>Pf of Th$^m$</u>: Let $\{G_\alpha\}$ be an open cover of $I$.



Sps no finite sub cover covers $I$

Subdivide $I$ into $2^k$ - $k$- cells whose union is $I$.

Then at least one cannot be covered by a finite sub-collection of $\{G_\alpha\}$. Call it $I_1$

Let

$$\delta = \left( \sum_{j=1}^{k} (b_j - a_j)^2 \right)^{1/2}$$

be the diagonal. diag $(I)$.

Then.

$$\text{diag}\,(I_1) = 2^{-1}\,\delta.$$

$$\Rightarrow \quad |\vec{x} - \vec{y}| < \delta/2 \quad \forall \; \vec{x}, \vec{y} \in I_1.$$

Iterate to obtain $k$-cells $\{I_n\}$ s.t

1) $I \supseteq I_1 \supseteq I_2 \cdots$

2) $I_n$ is NOT - covered by any finite subcollection of $\{G_\alpha\}$.

3). $|\vec{x} - \vec{y}| < 2^{-n}\delta \quad \forall \; \vec{x}, \vec{y} \in I_n$.

Lemma $\Longrightarrow \; \exists \; \vec{a} \in \bigcap_{n=1}^{\infty} I_n \cap I$

$\exists\, \alpha$ s.t $\quad \vec{a} \in G_\alpha$. since $I$ is covered by $G_\alpha$

$G_\alpha$ open $\Rightarrow$ $\exists$ $r>0$ s.t $B_r(\vec{a}) \subseteq G_\alpha$.

If $n$ big enough s.t $2^{-n}\delta < r$.

Then since $\vec{a} \in I_n$ for any $\vec{x} \in I_n$.

$$|\vec{x} - \vec{a}| < 2^{-n}\delta < r$$

$\Rightarrow$ $\vec{x} \in B_r(\vec{a}) \Rightarrow \vec{x} \in G_\alpha$.

$\Rightarrow$ $I_n \subset G_\alpha$ i.e $I_n$ is covered by <u>one</u> open set in $\{G_\alpha\}$.

Contradiction!

<u>Def$^n$</u> A set $A \subset X$ is called <u>bounded</u> (bdd.) if $\exists$ $R>0$ s.t $A \subset B_R(p)$ for some $p \in X$

<u>Th$^m$</u>: $K \subset \mathbb{R}^k$ The foll are equivalent

1) $K$ is closed & bounded

2) $K$ is compact

3) $K$ is limit point compact.

<u>Pf</u>: 1) $\Rightarrow$ 2) $K$ is bdd. $\Rightarrow$ $K \subset I$ for some $k$-cell. $\Rightarrow$ $K$ is compact since $I$ is compact & $K$ is closed.

2) $\Rightarrow$ 3) Already proved for general metric sp.

3) $\Rightarrow$ 1). Sps $K$ is not bounded $\Rightarrow \exists \{x_n\}$ in $K$ s.t $|x_n| > n \quad \forall n$.

Clearly $x_n$ has no limit point in $\mathbb{R}^k$ which contradicts l.p compactness.

<u>Claim</u>: $K$ closed.

<u>Pf</u>: Sps $p \in X$ is a l.p of $K$. Let $x_n \in B_{1/n}(p)$ s.t $x_n \neq p$. and $x_n \in K$. We can always pick such points since $p$ is a l.p.

<u>Fact</u> (Th$^m$ 2.20 in Rudin) If $p$ is a l.p of $K$. $B_r(p) \cap K$ has infinite points $\forall r > 0$.

$\Rightarrow$ Can make sure all $x_n$s are distinct and so $\{x_n\}$ is an infinite seq.

$\xRightarrow{\text{l.p compactness}} \exists$ l.p $x_0 \in K$ of $\{x_n\}$

<u>Claim</u>: $x_0 = p$ and hence $p \in K$. (so $K$ is closed).

<u>Pf</u>: For any $\varepsilon > 0$, $\exists \ \underline{N} > 2/\varepsilon$ s.t $x_N \in B_{\varepsilon/2}(x_0)$.

But $x_N \in B_{1/N}(p)$.

$$\Rightarrow d(p, x_0) \leq d(p, x_N) + d(x_0, x_N)$$

$$\leq 1/N + \frac{\varepsilon}{2} \leq \varepsilon.$$

$$\Rightarrow d(p, x_0) \leq \varepsilon \quad \forall \varepsilon > 0 \Rightarrow d(p, x_0) = 0$$

$$\Rightarrow p = x_0$$

• <u>Connected Sets</u>: $(X, d)$ metric space.

<u>Def$^n$</u>: $A, B \subseteq X$ are called <u>separated</u> if $A \cap \bar{B}$ and $B \cap \bar{A}$ are both empty

A set $E \subset X$ is called <u>connected</u> if $E$ is <u>not</u> the union of two non-empty separated sets.

<u>Example</u>: $\mathbb{Q} \subset \mathbb{R}$ is <u>disconnected</u>. Consider.

$$A = (-\infty, \sqrt{2}) \cap \mathbb{Q}$$
$$B = (\sqrt{2}, \infty) \cap \mathbb{Q}$$

Then $A \cup B = \mathbb{Q}$ but $A$ and $B$ are separated.

since $A \cap \bar{B} = (-\infty, \sqrt{2}) \cap [\sqrt{2}, \infty) \cap \mathbb{Q} = \emptyset$.

$B \cap \bar{A} = \emptyset$

<u>Th$^m$</u> $E \subset \mathbb{R}^1$ is connected if and only if whenever $x, y \in E$ and $x < z < y$, then $z \in E$.

<u>Pf</u>: $\Rightarrow$. Sps $x, y \in E$ and $z \notin E$, $x < z < y$. Then consider

$$A = E \cap (-\infty, z).$$
$$B = E \cap (z, \infty)$$

$A, B$ are non-empty since $x \in A$, $y \in B$.

Also $A \subset (-\infty, z)$, $B \subset (z, \infty)$.

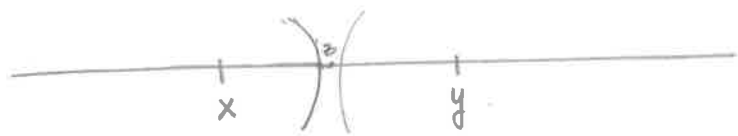
$\rightarrow \quad \bar{A} \subset (-\infty, z]$, $\bar{B} \subset [z, \infty)$.

$\Rightarrow \quad \bar{A} \cap B = B \cap \bar{A} = \emptyset$.

So $A, B$ separated. Clearly $E = A \cup B \Rightarrow$
$E$ disconnected. Contradiction!

$\Leftarrow$ Sps $E$ is disconnected. Then $\exists$ non-empty separated $A, B$ s.t. $E = A \cup B$. Let $x \in A$ and $y \in B$, and let

$$z = \sup(A \cap [x, y])$$




Clearly $z \in \bar{A} \Rightarrow z \notin B$ since $A, B$ are separated.
In particular $x \leq z < y$.

1) If $z \in A$ then $\exists$ $z \notin \bar{B} \Rightarrow \exists$ $z_1 \notin B$ s.t.

$$z < z_1 < y.$$

Since $z$ is sup, $z_1 \notin A \Rightarrow z_1 \notin E$.
So this contradicts property of $E$.

2) If $z \notin A$ then $z \notin E$ but $x < z < y$
contradicting property of $E$.

$\underline{\text{Th}^m}$: $(X, d)$ is connected if and only if there are no subsets $A \neq \phi, X$ which are <u>both</u> open and closed.

<u>Pf</u>: $\underline{\Rightarrow}$ Sps $A \subset X$ open & closed $\Rightarrow$ $B = A^c$ is also open & closed.

$$X = A \cup B.$$

$$A \cap \overline{B} = A \cap B = \phi \;, \; B \cap \overline{A} = \phi$$

$\Rightarrow$ A and B are separated.

But X is connected $\Rightarrow$ $A = \phi$ or $X$.

$\underline{\Leftarrow}$ Sps X is disconnected. Then $\exists$ A, B

$$X = A \cup B$$

$A \neq \phi$ and $B \neq \phi$ and $A \cap \overline{B} = B \cap \overline{A} = \phi$.

<u>Claim</u>: A and B are closed & open.

<u>Pf</u>: $B \cap \overline{A} = \phi \Rightarrow \overline{A} \subseteq A$ since $A \cup B = X$.

But $A \subseteq \overline{A} \Rightarrow A = \overline{A} \Rightarrow A$ is closed

IIIly B is closed.

But $A^c = B \Rightarrow$ A is open & B is open.

Contradiction! So X is connected.

Rk: (without proof). Recall that openness & closedness depend on the particular metric space. Compactness & Connectedness are intrinsic in the foll sense: Sps. $K \subset Y \subset X$ and $Y$ has the induced distance $d_Y$. Then $K$ is compact (resp. connected) in $(Y, d_Y)$ if and only if it is compact (resp conn.) in $(X, d)$.