MATH 104 - Lecture 1

- <u>Notation</u>:

$\mathbb{N} :=$ set of natural #s 0, 1, 2, 3, . . .

$\mathbb{Z} :=$ set of integers −2, −1, 0, 1, 2, . . .

$\mathbb{Q} := \{ p/q \mid p, q \in \mathbb{Z},\ (p, q) = 1 \}$.

g.c.d.

called the rationals.

- <u>Prop</u>. There is no $r \in \mathbb{Q}$ s.t $r^2 = 2$.

<u>Pf</u>. Proceed by <u>contradiction</u>. Sps $r = p/q$ s.t

$$r^2 = 2,$$

where $p, q \in \mathbb{Z}$, $(p, q) = 1$.

Then

$$p^2 = 2q^2.$$

$\Rightarrow$ 2 divides p^2 (denoted by $2 \mid p^2$).

2 is prime $\Rightarrow 2 \mid p$.

Sps $p = 2m \Rightarrow 4m^2 = 2q^2$

$\Rightarrow q^2 = 2m^2$.

Same reasoning $\Rightarrow 2 \mid q$.

$\Rightarrow$ 2 is a common factor. Contradiction!

Hence proof is complete.

- <u>$\mathbb{Q}$ is "incomplete"</u>. The sequence $\{r_n\}$.

$$1,\ 1.4 = \frac{7}{5},\ 1.41 = \frac{141}{100},\ \ldots$$

is s.t. $r_n^2 \text{"}\longrightarrow\text{"} 2$. So $r_n \text{"}\longrightarrow\text{"} x$ where x cannot be rational. So there are gaps in rationals.

- Note

$$A = \{ r \in \mathbb{Q} \mid r^2 < 2 \} \leftarrow \text{Dedekind Cuts.}$$

Claim: A has no "largest" element.

Pf: Let $r \in A$. Consider

$$s = r + \frac{2 - r^2}{2 + r}.$$

Then $s > r$.

Also $2 - s^2 = 2\left(\frac{2 - r^2}{(2 + r)^2}\right) > 0$.

So $s \in A$, $s > r \Rightarrow A$ has no largest element.

One can attempt to complete $\mathbb{Q}$ by "filling" in least upper bounds.

- Set notation:

$x \in A$:= x is an element of A.

$x \notin A$:= x is not " " " "

ϕ := empty set.

Subset $A \subseteq B$ = $x \in A \Rightarrow x \in B$.

$\Leftrightarrow B \supseteq A \leftarrow$ Superset

$\phi \subseteq A$, $A \subseteq A$ $\forall A$.

Proper Subset : $A \subsetneq B$.

$A = B \iff A \subseteq B$ AND $B \subseteq A$.

- $\exists$:= there exists, $\forall$: for all,
- <u>ORDERED SETS</u> : Let S be a set.

<u>Def^n</u>: An <u>order</u> on S is a relation, denoted by $\leqslant$. s.t

1) $x, y \in S$ then <u>exactly</u> one of

$$x \leq y, \quad \cancel{x = y}, \quad y \leq x.$$

holds

2) Transitivity : $x, y, z \in S$.

$$x \leq y, \quad y \leq z \implies x \leq z.$$

3) Reflexivity : $x \leq x$.

4) Anti-symmetry $x \leq y$ AND $y \leq x \Rightarrow x = y$.

We ~~say $x \leq y$ to mean either $x = y$ or $x < y$~~.

<u>Example</u> $\mathbb{Q}$: For $r, s \in \mathbb{Q}$, we say $r \leq s$ if $s - r$ is a ~~+ve~~ non-negat rational.

Say $r = s$ if $s - r = 0$.

<u>Def^n</u>: (Upper bound) : Sps S is ordered and $E \subset S$. If $\exists$ $\beta \in S$ s.t $x \leq \beta$ (resp. $x \geqslant \beta$) $\forall$ $x \in E$, we say E is <u>bounded above</u> (resp. below), and β is an <u>upper bound</u> (resp. lower bou

Rk: 1) Upper bounds need not be unique.

2) We can similarly define lower bounds.

• Defn. (Least Upper bound / Supremum). S ordered. $E \subset S$ bdd. above. Sps $\exists\ \alpha \in S$ s.t.

(1) $x \leq \alpha \quad \forall\ x \in E$.

(2) $\gamma < \alpha \Rightarrow \exists\ x \in E$ s.t. $x > \gamma$.

Then α is called the least upper bound (l.u.b), an we write

$$\alpha = \sup E.$$

Similarly define g.l.b, and denote

$$\beta = \inf E.$$

Example: 1) $A = \{p \in \mathbb{Q} \mid p^2 < 2\}$.

Then there is no rational l.u.b. of A.

2) $E = \{1/n \mid n = 1, 2, \ldots\} \subset \mathbb{Q}$.

$$\sup E = 1 \in E$$
$$\inf E = 0 \notin E.$$

Defn. S is said to have l.u.b property if $\forall\ E \subset S$, $E \neq \phi$ and E bdd above; then $\sup E$ exist in S.

Similarly one can define g.l.b property

$\underline{\text{Th}^m}$: Sps S has l.u.b property. Let $B \subset S$, $B \neq \phi$ and bounded below. Then $\inf B$ exist in S.

$\underline{\text{Pf}}$: Let $L = \{\beta \in S \mid \beta \leq x \quad \forall\, x \in B\}$.

$L \neq \phi$ since $B \neq \phi$ & bdd below.

L is bounded above, infact any $x \in B$ is an u.b.

$\Rightarrow \beta_0 = \sup L$ exists in S.

$\underline{\text{Claim}}$: $\beta_0 = \inf B$.

$\underline{\text{Pf}}$: (1). $\beta_0 \leq x \quad \forall\, x \in B$

If not, then $\exists\, x_0 \in B$ s.t $x_0 < \beta_0$.

$\Rightarrow x_0$ is not an u.b for L. $\Rightarrow x_0 \notin B$ contradiction

$\Rightarrow x_0$ is not a lower bound for B.

(2) $\beta > \beta_0 \Rightarrow \beta$ is not a lower bound for B.

Self evident. If it is then $\beta \in L$.

But $\beta > \beta_0 = \sup L$

contradiction.

Stuff:

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- GRADING:

  Assig~~nm~~nments (20%): 12 ~~Assignments~~ Best 10/12.

  Mid~~e~~-Terms (40%) 2 of 40 points each.

  No make-up.

  Mid Term Score = Best 2 of (MT-1, MT-2, F/2).

  Final Exam (40%): 80 points

  CANNOT PASS IF YOU MISS FINAL.

NOTE: For M-T, if you are ~~attending~~ representing uni & that clashes with any of MT, let me know BEFORE Jan 31.

- Outline of course:
  1. Real & Complex Numbers
  2. Metric Spaces.
  3. Seq / Series.
  4. Limits, Cont.
  5. Diff.
  6. Integration
  7. Seq^{ns} of functions
  8. Power Series / Taylor Expansion.

MATH 104: Lecture 2

- **Recall**:

1) S ordered, has the l.u.b. property if $\forall E \subseteq S$, non empty & upper bounded has $\sup E$.
2) $\mathbb{Q}$ does NOT have this property

e.g. $A = \{ r \in \mathbb{Q} \mid r^2 < 2 \}$.

- **Fields**:

Def^n: A field is a set F with two binary operations addition $(+)$ and multiplication $(\cdot)$ satisfying.

(A) Addition axioms.

A1 $x, y \in F \Rightarrow x + y \in F$

A2 (Commutativity): $x + y = y + x$

A3 (Associativity): $x + (y + z) = (x + y) + z$.

A4 (Identity): A zero $0 \in F$ s.t. $x + 0 = x \quad \forall x \in F$.

A5 (Inverse) $\forall x \in F$, $\exists -x \in F$ s.t.

$$x + (-x) = 0.$$

(M) Multiplication axioms:

M1 $x, y \in F \Rightarrow x \cdot y \in F$.

M2 ($x \cdot y = y \cdot x \quad \forall x, y$.

M3 $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.

M4. $\exists\ 1 \in F,\ 1 \neq 0$ s.t. $1 \cdot x = x \quad \forall\ x \in F$.

M5 If $x \in F$, $x \neq 0$, $\exists$ element, denoted $1/x \in F$

s.t. $$x \cdot \left(\frac{1}{x}\right) = 1.$$

(D) Distributive axiom $\forall\ x, y, z \in F$

$$x \cdot (y + z) = x \cdot y + x \cdot z.$$

Example: $\mathbb{Q}$ is a field.

• Some consequences: 1) $x + y = x + z \Rightarrow y = z$.

2) $x + y = x \Rightarrow y = 0$

3) $x \cdot 0 = 0$.

4) $-(-x) = x$.

5) If -1 is add inv of $1 \Rightarrow -x = -1 \cdot x$.

Pf of 5: $x + -1 \cdot x = x(1 + -1) = x \cdot 0 = 0$.

• Def^n: A ordered field is a field F with an order $(\leq)$ s.t

(1) $x + y \leq x + z \iff y \leq z$.

(2) $xy \geq 0$ if $x, y \in F$ & $x, y \geq 0$.

We call x positive (resp. negative) if $x > 0$ (resp. < 0).

• Consequences: 1) $x > 0 \Rightarrow -x < 0$ and vice-versa

2) $x > 0,\ y < z \Longrightarrow xy < xz$

3) $x \neq 0 \Rightarrow x^2 > 0$, In particular $1 \neq 0$

4) $0 < x < y \Rightarrow 0 < 1/y < 1/x$

Pf 3) $x > 0 \overset{\text{Axioms}}{\Longrightarrow} x \cdot x > 0$. i.e $x^2 > 0$.

$x < 0 \overset{1)}{\Longrightarrow} (-x) > 0 \Rightarrow (-x)(-x) > 0$

$\Rightarrow x^2 > 0$.

4) By 1) $yv \leq 0$ if $y > 0,\ v \leq 0$.

But $y \cdot y^{-1} = 1 > 0 \Rightarrow y^{-1} > 0$. Similarly $x^{-1} > 0$.

Then $y \cdot x^{-1} > x \cdot x^{-1} = 1$

$\Rightarrow y^{-1} \cdot y \cdot x^{-1} > y^{-1} \Rightarrow x^{-1} > y^{-1}$

• The Real field:

Th^m: There exists a unique ordered field $\mathbb{R}$ which has the least upper bound property & has $\mathbb{Q}$ as a sub-field.

Def^n: The above field is called the field of real numbers.

Rk: $\mathbb{Q} \subseteq \mathbb{R}$ means that when $+, \cdot, \leq$ are restricted from $\mathbb{R}$ to $\mathbb{Q}$, we get the usual addition, multipl. and order on $\mathbb{Q}$.

Construction :

Def^n: A cut is a subset $\alpha \subset \mathbb{Q}$ s.t

1) $\alpha \neq \emptyset$ or $\mathbb{Q}$.
2) $p \in \alpha$, $q \in \mathbb{Q}$, $q < p \Rightarrow q \in \alpha$.
3) $p \in \alpha \Rightarrow \exists\ r \in \alpha$ s.t $p < r$.

Eg: 1) $\alpha = \{ r \in \mathbb{Q}_+ \mid r^2 < 2 \} \cup \{0\} \cup \{\mathbb{Q}_-\}$

2) For $p \in \mathbb{Q}$ we can associate a unique cut

$$p^* := \{ r \in \mathbb{Q} \mid r < p \}$$

Define

$$\mathbb{R} := \{ \alpha \subset \mathbb{Q} \mid \alpha \text{ a cut} \}$$

We define

(1) Order: $\alpha < \beta$ if α is a proper subset of β.

(2) Addition:

$$\alpha + \beta = \{ r + s \mid r \in \alpha,\ s \in \beta \}.$$

(3) Define identity to be 0^*

Define for $\alpha \in \mathbb{R}$

$$-\alpha = \{ p \mid \exists\ r \text{ s.t } -p - r \notin \alpha \}$$

e.g.: Let $\alpha = 2^*$. Then $\alpha = \{p \in \mathbb{Q} \mid p < 2\}$. Then

$$-\alpha = \{p \in \mathbb{Q} \mid \exists\, r > 0 \text{ s.t. } -p - r \notin \alpha\}$$

$$= \{p \in \mathbb{Q} \mid \exists\, r > 0, \text{ s.t. } -p - r \geq 2\}$$

$$= \{p \in \mathbb{Q} \mid \exists\, r > 0 \text{ s.t. } p \leq -2 - r\}$$

$$= \{p \in \mathbb{Q} \mid p < -2\} = (-2)^*$$

• Multiplication: Let $\mathbb{R}_+ = \{\alpha \in \mathbb{R} \mid \alpha > 0^*\}$.

For $\alpha \in \mathbb{R}^*_+$, $\beta \in \mathbb{R}_+$, define

$$\alpha \cdot \beta = \{p \in \mathbb{Q} \mid p \leq rs, \text{ for some } r \in \alpha, s \in \beta, r, s > 0\}$$

e.g.: $\alpha = 2^*$, $\beta = 3^*$. Then $\sup\{rs \mid r, s > 0, r \in 2^*, s \in 3^*\}$

$= 6$

$$\Rightarrow \quad \alpha \cdot \beta = \{p \mid p \leq 6\} = 6^* = (2 \cdot 3)^*$$

More generally, let $\alpha \cdot 0^* = 0^* \cdot \alpha = 0^*$ &

$$\alpha\beta = \begin{cases} (-\alpha)\cdot(-\beta) & \alpha, \beta < 0^* \\ -[(-\alpha)\cdot(\beta)], & \alpha < 0^*, \beta > 0^* \\ -[(\alpha)\cdot(-\beta)], & \alpha > 0^*, \beta < 0^* \end{cases}$$

1^* is the identity

For $\alpha \in \mathbb{R}_+$, define

$$\frac{1}{\alpha} = \{ p \in \mathbb{Q} \mid \exists\ r > 1 \text{ s.t } \frac{1}{pr} \notin \alpha \}.$$

e.g: $\alpha = 2^*$ Then

$$\frac{1}{2^*} = \{ p \in \mathbb{Q} \mid \exists\ r > 1 \text{ s.t } \frac{1}{pr} \notin (2)^* \}.$$

$$= \{ p \in \mathbb{Q} \mid \exists\ r > 1 \text{ s.t } \frac{1}{p \cdot r} \geq 2. \}$$

$$= \{ p \in \mathbb{Q} \mid \exists\ r > 1 \text{ s.t } p \leq \frac{1}{2r} \}$$

- If $\alpha < 0$, define $1/\alpha = -1/(-\alpha)$.

Th^m: (a) If $x, y \in \mathbb{R}$, $x > 0$, then $\exists\ n \in \mathbb{N}$, s.t

$$nx > y.$$

(b) If $x, y \in \mathbb{R}$, $x < y$. $\exists\ r \in \mathbb{Q}$ s.t

$$x < r < y.$$

Pf: (b). Since $y - x > 0$, $\exists\ n \in \mathbb{N}$ s.t $n(y-x) > 1$.

Claim. $\exists\ m \in \mathbb{Z}$ s.t $m - 1 < nx < m$

Pf: $\left.\begin{array}{l}\exists\ m_1 \in \mathbb{N} \text{ s.t } m_1 > -nx. \\ \exists\ m_2 \in \mathbb{N} \text{ s.t } m_2 > nx.\end{array}\right\} \Rightarrow -m_1 < nx < m_2.$

$\Rightarrow$ $\exists\ m$ s.t $m - 1 \leq nx < m$.

Then $nx < m \leq 1 + nx < ny$

$\Rightarrow x < \frac{m}{n} < y$. Put $r = m/n$.

MATH 104 : Lecture 3

- <u>Th^m</u>: (a) (Archimedean property) If $x, y \in \mathbb{R}$, $x > 0$, $\exists\, n \in \mathbb{N}$ s.t

$$nx > y.$$

(b) (Density of $\mathbb{Q}$) If $x, y \in \mathbb{R}$, $x < y$, $\exists\, r \in \mathbb{Q}$ s.t

$$x < r < y.$$

<u>Pf</u>: (b) $y - x > 0 \overset{(a)}{\Longrightarrow} \exists\, n$ s.t

$$n(y - x) > 1. \Rightarrow nx + 1 < ny.$$

<u>Claim</u>: $\exists\, m \in \mathbb{Z}$ s.t $m - 1 \le nx < m$

Assuming claim

$$nx < m \le nx + 1 < ny.$$

$$\Rightarrow x < \frac{m}{n} < y, \text{ put } r = m/n.$$

<u>Pf of Claim</u>: (a) $\Rightarrow \exists\, m_1$ s.t $m_1 > nx$.
$\exists\, m_2$ s.t $m_2 > -nx$.

$$\Rightarrow \quad -m_2 < nx < m_1.$$

If $m_1 - 1 \le nx$, put $m = m_1$, else $-m_2 < nx < m_1 - 1$

and keep going.

This proves the claim

- <u>Decimal Expansion</u>: Sps $x > 0$, $x \in \mathbb{R}$.
Let $n_0 \in \mathbb{N}$ be the largest integer s.t $n_0 \le x$.

Having chosen $n_0, n_1, \ldots, n_{k-1}$, let n_k be the largest integer s.t

$$n_0 + \frac{n_1}{10} + \frac{n_2}{10^2} + \cdots \frac{n_{k-1}}{10^{k-1}} + \frac{n_k}{10^k} \leq x.$$

Let

$$E = \left\{ n_0 + \frac{n_1}{10} + \cdots + \frac{n_k}{10^k} \;\middle|\; k = 0, 1, \ldots \right\}.$$

Claim: $x = \sup E$.

We denote

$$x = n_0 \cdot n_1 n_2 n_3 \ldots$$

e.g: $\sqrt{2} = 1.414\ldots\ldots$

- n^{th} roots:

Thm: For every $x > 0$, and every $n \in \mathbb{N}$, $\exists!$ unique $y \in \mathbb{R}$ s.t $\quad y^n = x$. (we write $y = \sqrt[n]{x} = x^{1/n}$).

Outline of proof: Uniqueness since $y_1 < y_2 \Rightarrow y_1^n < y_2^n$.

Let

$$E = \{ t > 0 \mid t^n < x \}.$$

If $t_0 = x/x+1$, then $t \in E \Rightarrow E$ is non-empty.

If $t > x+1 \Rightarrow t^{n-1} > 1 \Rightarrow t^n > t > x+1 > x$

$\Rightarrow t^n > x \Rightarrow t \notin E \Rightarrow x+1$ u.b for E

$\Rightarrow \sup E$ exist

Claim: $y = \sup E \rightarrow y^n = x$.

For proof, see Rudin.

· Complex numbers:

Def^n: We define

$$\mathbb{C} = \{(a,b) \mid a \in \mathbb{R},\ b \in \mathbb{R}\}.$$

with

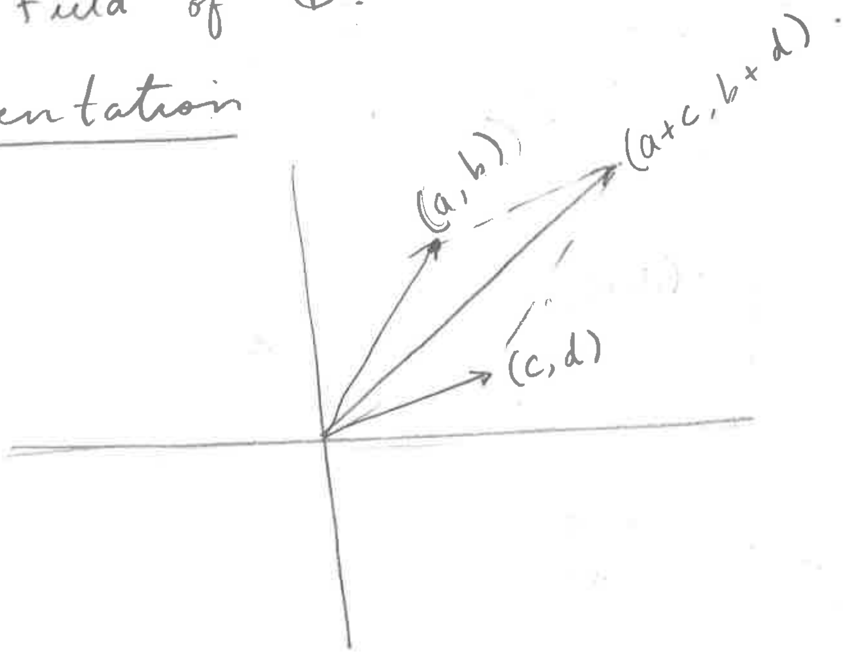
$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2,\ b_1 + b_2).$$

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1 a_2 - b_1 b_2,\ a_1 b_2 + a_2 b_1).$$

Prop: $(\mathbb{C}, +, \cdot)$ is a field with $(0,0)$ as the additive identity & $(1,0)$ as the multiplicat[ive] identity.

Rk: $\mathbb{R} \subset \mathbb{C}$ by considering $(a,0)$. This makes $\mathbb{R}$ as sub-field of $\mathbb{C}$.

· Geometric representation



Addition is equivalent to vector addition

Def^n $i = (0,1)$.

Then ① $i^2 = -1$.

② Any $z = (a,b)$ can be written as

$$z = a + ib.$$

Multiplication is by simple distribution

$$(a_1 + ib_1)(a_2 + i b_2) = a_1 a_2 + i^2 b_1 b_2 + i b_1 a_2 + i a_1 b_2$$
$$= (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1).$$

Def^n: $z = a + ib$, we say

$$a = Re(z), \quad b = Im(z).$$

Def^n (Conjugate). $z = a + ib$, we define

$$\bar{z} = a - ib.$$

Properties 1) $\overline{z + w} = \bar{z} + \bar{w}$

2) $\overline{zw} = \bar{z} \cdot \bar{w}$

3) $Re\, z = \dfrac{z + \bar{z}}{2}$, $Im\, z = \dfrac{z - \bar{z}}{2i}$.

(4) $z\bar{z}$ is real and positive

Pf of 4: $z\bar{z} = (a+ib)(a-ib)$

$$= a^2 + b^2 > 0.$$

· <u>Absolute Value function</u>

For $x \in \mathbb{R}$ we define

$$|x| = \begin{cases} x & ; \quad x \geq 0 \\ -x & \quad x < 0 \end{cases}$$

Then $|x| \geq 0 \ \forall \ x \in \mathbb{R}$.

<u>Note</u>: $|x|^2 = x^2 \implies |x| = \sqrt{x^2}$

For $z \in \mathbb{C}$, we define

$\boxed{|z| = \sqrt{z\bar{z}}}$. If $z = a + ib$, $\boxed{|z| = \sqrt{a^2 + b^2}}$

This has the important property that

(Triangle ineq) $\quad |z + w| \leq |z| + |w|$.

$|z|$ is the "magnitude" of the vector represented by z.

· <u>Finite, Countable, Uncountable Sets</u>

<u>Def^n</u>: Given two sets A, B, a <u>function</u> (or map) $f : A \to B$ is a rule to assign a <u>unique</u> element $f(x) \in B$ to every element $x \in A$.

We say 1) One-One (injective)

$$f(x) = f(y) \Rightarrow x = y.$$

2) Onto (Surjective) $\forall \ y \in B, \exists \ x \in A$ s.t

$$f(x) = y.$$

y is called image & x pre-image.

3) Bijective if f is one-one & onto.

Defn: If there is a bijection from A to B, we say A & B have the same cardinality and denote $A \sim B$.

This is an equivalence relation in the sense that

(a) Reflexive $A \sim A$

(b) Symmetric $A \sim B \Leftrightarrow B \sim A$.

(c) Transitive $A \sim B$, $B \sim C \Rightarrow A \sim C$.

Defn: For any $n \in \mathbb{N}_+$ denote

$$J_n = \{1, 2, \dots n\}.$$

$$J = \mathbb{N} \setminus \{0\}.$$

For a set A, we say.

(a) A is finite if $A \sim J_n$.

(b) A is infinite if not finite

(c) A is countable if $A \sim J$

(d) A is uncountable if not countable.

Rk: ① If $A \sim J_n$, then A has n-elements.

② $\mathbb{Z}$ is countable. We can define $f: J \to \mathbb{Z}$

$\mathbb{Z}$	0,	-1,	$\cancel{1}$,	-2,	2,	...
J	1	2	3	4	5,	...

i.e.
$$f(n) = \begin{cases} \frac{n}{2} & n \text{ even} \\ -\left(\frac{n-1}{2}\right) & n \text{ odd} \end{cases}$$

Def^n: By a seq in S we mean a function defined from $f : J \to S$, for some set S. If

$$x_n = f(n).$$

we write the seq as $\{x_n\}_{n=1}^{\infty}$.

Rk: Every countable set can be regarded as the range of a sequence.

Th^m: Every infinite subset of a countable set S is also countable.

Pf: Sps $E \subset S$, E infinite. Let $S = \{x_n\}_{n=1}^{\infty}$. We choose a sub-seq in the foll way - Let n_1 be the smallest +ve integer s.t $x_{n_1} \in E$. Once $n_1, n_2, \dots n_{k-1}$ are picked, we let n_k be the smallest integer greater than n_{k-1} s.t $x_{n_k} \in E$.

Put $$f(k) = x_{n_k}, \quad f : J \to E.$$

Clearly f is both 1-1 and onto E

$\Rightarrow$ E is countable.

~~<u>Thm</u>: Every infinite subset of a countable set A is countable.~~

~~<u>Pf</u>: Let $E \subset A$, E infinite~~