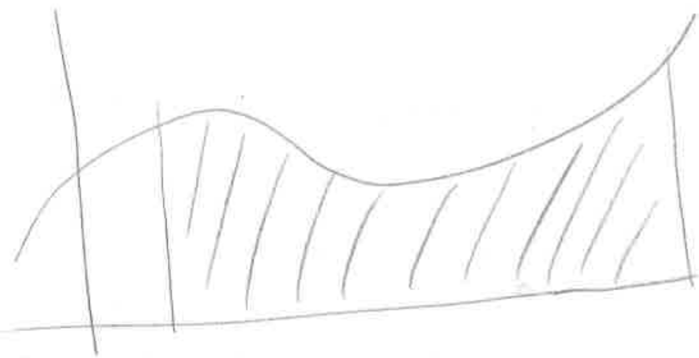
6. INTEGRATION

· Given $f : [a,b] \to \mathbb{R}$,

$$\int_a^b f(x)\,dx$$

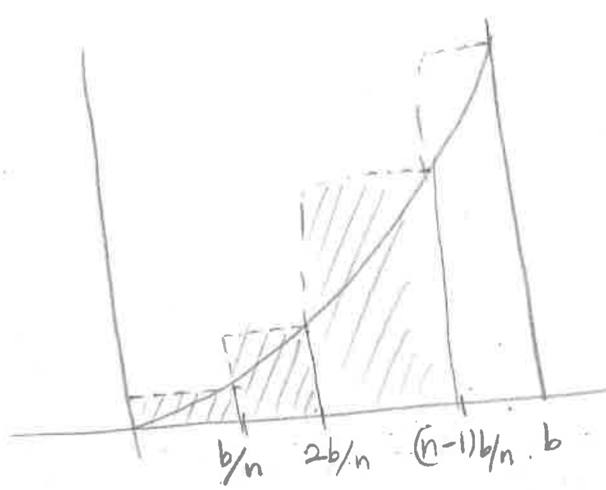
compute the area under $y = f(x)$ from $x = a$ to $x = b$.



To understand the difficulties involved, consider the foll examples

· 2 Examples:

1) $f(x) = x^2$. Area from $t = 0$ to $\cancel{t} = a$.



We use the so-called right-end approx

Consider partition

$$0 = t_0 < t_1 = \frac{b}{n} \cdots < t_k = \frac{kb}{n} < \cdots < t_n = \frac{n \cdot b}{n} = b.$$

We consider rectangle $[t_{k-1}, t_k] \times [0, f(t_k)] = R_{k,n}$.
Let the sum of areas be A_n. Then
it is reasonable that

$$\text{Area under } y = f(t) \text{ from } 0 \text{ to } b = A = \lim_{n \to \infty} A_n$$

Clearly $\text{Area}(R_{k,n}) = (t_k - t_{k-1}) \cdot f(t_k)$.

$$= \frac{b}{n} \cdot t_k^2 = \frac{b}{n} \cdot \frac{k^2 b^2}{n^2} = \frac{b^3}{n^3} \cdot k^2$$

$$\Rightarrow \quad A_n = \frac{b^3}{n^3} \sum_{k=1}^{n} k^2$$

<u>Claim</u> $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

<u>Pf</u>: $(k+1)^3 - k^3 = 1 + 3k + 3k^2$

Summing from $k=1$ to n.

$$(n+1)^3 - 1 = n + 3\sum_{k=1}^{n} k + 3S, \quad S = \sum_{k=1}^{n} k^2$$

$$\Rightarrow n^3 + 3n^2 + 3n = n + \frac{3n(n+1)}{2} + 3S.$$

$$\Rightarrow \quad 3S = n\left[n^2 + 3n + 3 - \frac{3(n+1)}{2} - 1\right]$$

$$\Rightarrow \quad 3S = \frac{n(n+1)(2n+1)}{2} \qquad \text{Done!}$$

So,

$$A_n = b^3 \left[\frac{(n+1)(2n+1)}{6n^2}\right]$$

$$= \frac{b^3}{6}(1 + 1/n)(2 + 1/n)$$

$$\xrightarrow{n \to \infty} \frac{b^3}{6} \cdot 2 = \frac{b^3}{3}$$

So $A = \frac{b^3}{3}$.

We write this as $\int_0^b t^2\,dt = b^3/3$.

2) Now consider

$$f(t) = \begin{cases} 1 & t \in \mathbb{Q} \\ 0 & t \notin \mathbb{Q} \end{cases}$$

We compute $\int_0^1 f(t)\,dt$ & $\int_0^{\sqrt{2}} f(t)\,dt$ using right hand approx.

For $\int_0^1 f(t)\,dt$ Consider partition $0 = t_0 < t_1 = 1/n < \dots < t_n = 1$

Now for any k, $t_k \in \mathbb{Q} \Rightarrow f(t_k) = 1$.

$\Rightarrow$ So each rectangle is of height one & width $1/n$.

$\Rightarrow \quad A_n = n \cdot \frac{1}{n} = 1.$

$\Rightarrow \quad$ Area $= 1.$

- For $\int_0^{\sqrt{2}} f(t)\,dt$ Again consider partition

$$0 = t_0 < t_1 = \frac{\sqrt{2}}{n} < \; < t_k = \frac{\sqrt{2}}{n} \cdot k < \cdots < t_n = \sqrt{2}.$$

Here $t_k \notin \mathbb{Q} \; \forall k \Rightarrow f(t_k) = 0 \; \forall k.$

So right hand approx $\Rightarrow$

Area $= 0.$

This is a problem, since $[0,1] \subset [0, \sqrt{2}]$ & $f \geq 0.$

So we would expect area to increase.

- Riemann Integrability

Def^n: Let $[a,b]$ be an interval. By a partition $\mathcal{P}$ we mean a finite set

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b.$$

We write

$$\Delta t_k = t_k - t_{k-1}, \quad k = 1, 2, \cdots, n.$$

Sps $f : [a,b] \to \mathbb{R}$ is bounded, & $\mathcal{P}$ is a partition. We set

$$M_k = \sup_{t \in [t_{k-1}, t_k]} f(t)$$

$$m_k = \inf_{[t_{k-1}, t_k]} f(t).$$

$$U(P,f) = \sum_{k=1}^{n} M_k \Delta t_k$$

$$L(P,f) = \sum_{k=1}^{n} m_k \Delta t_k.$$

Sps $m < f(t) < M$ for $t \in [a,b]$. Then

$$m(b-a) < L(P,f) < U(P,f) < M(b-a).$$

So, $L(P,f)$ & $U(P,f)$ are <u>bounded</u> as we vary P.

<u>Def^n</u>: We define the <u>upper</u> <u>Riemann</u> <u>integral</u> of f by

$$\overline{\int_a^b} f(t)\,dt = \inf_P U(P,f)$$

& the <u>lower</u> <u>Riemann integral</u> of f by

$$\underline{\int_a^b} f(t)\,dt = \sup_P L(P,f).$$

We say f is <u>Riemann integrable</u> if both the values are equal & we then write

$$\int_a^b f(t)\,dt := \overline{\int_a^b} f(t)\,dt = \underline{\int_a^b} f(t)\,dt.$$

<u>Example</u>: Again, consider

$$f(t) = \begin{cases} 1 & t \in \mathbb{Q} \\ 0 & t \notin \mathbb{Q} . \end{cases}$$

& let P be <u>any</u> partition of $[a,b]$

$$a = t_0 < t_1 \ldots < t_n = b .$$

Since any $[t_{k-1}, t_k]$ contains both rationals & irrationals,

$$M_k = 1, \quad m_k = 0 \quad \forall k .$$

$$\Rightarrow \quad U(P,f) = (b-a), \quad L(P,f) = 0 .$$

~~This is true~~ for any P. So

$$\overline{\int_a^b} f(t)\,dt = (b-a), \quad \underline{\int_a^b} f(t)\,dt = 0$$

So, f is <u>NOT</u> Riemann integrable.

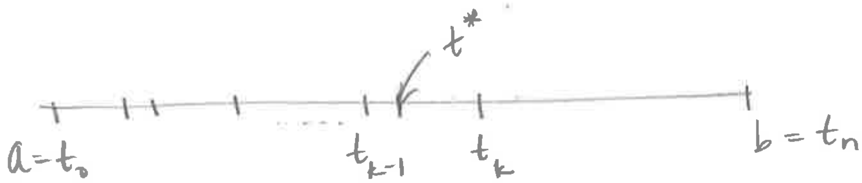
• <u>Criteria for integrability</u>. Let $f: [a,b] \to \mathbb{R}$ <u>bounded</u>.

<u>Def</u>^{n}: Given partitions P & P^* of $[a,b]$, we say P^* is a <u>refinement</u> of P if $P^* \supset P$.

Th^m 6.1: If P^* is a refinement of P, then

$$L(P,f) \le L(P^*,f)$$
$$U(P^*,f) \le U(P,f).$$

Pf: Let's prove the 1st inequality. Sps P^* has only one extra point than P. Call this point t^*. Let $P = \{t_0, \dots, t_n\}$ & sps $t^* \in (t_{k-1}, t_k)$.



Set
$$W_1 = \inf_{[t_{k-1}, t^*]} f(t), \quad W_2 = \inf_{[t^*, t_k]} f(t).$$
$$m_i = \inf_{[t_{i-1}, t_i]} f(t).$$

Clearly $W_1, W_2 \ge m_k$.

$$L(P,f) = \sum_{i=0}^{n} m_i \Delta t_i, \quad L(P^*,f) = \sum_{i=0}^{k-1} m_i \Delta t_i + \sum_{i=k+1}^{n} m_i \Delta t_i$$
$$+ W_1(t^* - t_{k-1}) + W_2(t_k - t^*).$$

$$\Rightarrow L(P^*,f) - L(P,f) = W_1(t^* - t_{k-1}) + W_2(t_k - t^*)$$
$$- m_k(t_k - t_{k-1})$$
$$\ge m_k(t^* - t_{k-1}) + m_k(t_k - t^*)$$
$$- m_k(t_k - t_{k-1}) \ge 0.$$

If P^* has m points more than P, apply the reasoning m times.

The 2^{nd} inequality is also proven similarly.

<u>Th^m 6.2</u>: $f \in R[a,b]$ if and only if $\forall\, \varepsilon > 0$, $\exists$ partition P s.t

$$U(P,f) - L(P,f) < \varepsilon$$

<u>Pf</u>: $\Rightarrow$ Sps $f \in R[a,b]$. So,

$$\sup_P L(P,f) =: \underline{\int}_b^a f(t)\,dt = \overline{\int}_a^b f(t)\,dt := \inf_P U(P,f)$$

Given $\varepsilon > 0$, $\exists$ partition P_1 s.t

$$\int_a^b f(t)\,dt - L(P_1,f) < \frac{\varepsilon}{2}$$

$\exists$ P_2 s.t

$$U(P_2,f) - \int_a^b f(t)\,dt < \frac{\varepsilon}{2}$$

Consider $P^* = P_1 \cup P_2$. Then P^* <u>refinement</u> of P_1 & P_2.

So
$$\int_a^b f(t)\,dt - L(P^*,f) < \int_a^b f(t)\,dt - L(P_1,f) < \frac{\varepsilon}{2}$$

Similarly $U(P^*,f) - \int_a^b f(t)\,dt < \frac{\varepsilon}{2}$

$$\Rightarrow \quad U(P^*,f) - L(P^*,f) < \varepsilon$$

$\Leftarrow$ For every partition P

$$L(P,f) \le \underline{\int_a^b} f(t)\,dt \le \overline{\int_a^b} f(t)\,dt \le U(P,f).$$

$\Rightarrow$

Given $\varepsilon > 0$, one can make

$$0 \le \overline{\int_a^b} f(t)\,dt - \underline{\int_a^b} f(t)\,dt \le \varepsilon.$$

So the 2 quantities have to be the same

$\Rightarrow f \in \mathcal{R}[a,b]$

• Continuity & Integrability

Def^n: A subset $A \subset [a,b]$ is said to be of measure zero if $\forall\, \varepsilon > 0$, $\exists$ at most finitely many open intervals $I_n = (a_n, b_n)$ s.t

(1) $A \subset \bigcup_{n=1}^{N} I_n$ (2) $\sum_{n=1}^{N} (b_n - a_n) \le \varepsilon$.

Examples 1) Countable Set: $A = \{p_n\}$. Let $\varepsilon > 0$. Consider $I_n = (p_n - \varepsilon/2^{n+1}, p_n + \varepsilon/2^{n+1})$. length$(I_n) = \varepsilon/2^n$. Also let $K := \bigcup (p_n - \varepsilon/2^{n+2}, p_n + \varepsilon/2^{n+2})$. Then $[a,b]$ is covered by $\{I_n\}$ & K^c. Being compact & since no $p_n \notin K$ $\exists\, n_1, n_2, \dots, n_N$ s.t $\{p_n\} \subset \bigcup_{k=1}^{N} I_{n_k}$ & $\sum$ length$(I_n) < \varepsilon$

2) Cantor Set: $E_0 = [0,1]$, $E_1 = [0, 1/3] \cup [2/3, 1]$ i.e remove $(1/3, 2/3)$ from E_0.

$E_2 = [0, 1/9] \cup [2/9, 3/9] \cup [6/9, 7/9] \cup [8/9, 1]$

So we get $\{E_n\}$ s.t

(1) $E_1 \supset E_2 \supset E_3 \supset \cdots$

(2) E_n is the union of 2^n intervals, each of length 3^{-n}.

The <u>Cantor Set</u> is defined by

$$\mathcal{C} = \bigcap_{n=1}^{\infty} E_n$$

<u>Claim</u> $\mathcal{C}$ is of measure zero.

<u>Pf</u>: Let $\varepsilon > 0$. Pick N s.t $(2/3)^N < \varepsilon$

$E_N = \bigcup_{k=1}^{2^N} I_k$, where each $\ell(I_k) = 3^{-N}$

So $\mathcal{C} \subset \bigcup_{k=1}^{2^N} I_k$, $\sum_{k=1}^{2^N} \ell(I_k) = (2/3)^N < \varepsilon$

<u>Thm 6.3</u>: Let $f: [a,b] \to \mathbb{R}$ bounded. If the set of discontinuities of f has measure zero, then $f \in R[a,b]$

<u>Cor</u> 6.4. If $f: [a,b] \to \mathbb{R}$ has at most countably many discontinuities $\Longrightarrow f \in R[a,b]$. In <u>particular</u> a cont. function is always integrable

Pf: Let $\varepsilon > 0$. Let $A \subset [a,b]$ be the set of discont.

For any $\varepsilon' > 0$ $A \subset \bigcup_{n=1}^{N} (a_n, b_n)$ s.t $\sum (b_n - a_n) < \varepsilon'$. (*)

Sps $|f(t)| < M$ $\forall$ $t \in [a,b]$.

We'll pick ε' later, to depend on ε.

Now, let $K = [a,b] \setminus \bigcup_{n=1}^{N} (a_n, b_n)$

Then K is closed (and of course bdd.) $\Rightarrow$ K is compact.

f is cont. on K $\Rightarrow$ f is uniformly cont on K.

So $\exists$ δ s.t

$|s-t| < \delta$, $s, t \in K$ $\Rightarrow$ $|f(s) - f(t)| < \varepsilon'$. (* *)

Consider the partition $P = \{t_0, \dots, t_n\}$ s.t

1) $a_n, b_n \in P$ $\forall$ $n = 1, 2, \dots, N$.

2) No point in (a_n, b_n) belongs to P.

3) $t_{k-1} \neq a_n$ (& so $x_k \neq b_n$) $\Rightarrow$ $\Delta x_k < \delta$.

Note that $M_k - m_k \le 2M$ $\forall$ k.

(* *) $\Rightarrow$ $M_k - m_k \le \varepsilon'$ if $t_{k-1} \neq a_n$

$$U(P,f) - L(P,f) = \sum_k (M_k - m_k) \Delta t_k$$

$$= \sum_{t_{k-1} \neq a_n} (M_k - m_k) \Delta t_k + \sum_{t_{k-1} = a_n} (M_k - m_k)(b_n - a_n).$$

$$\leq \varepsilon' \sum_{t_{k-1} \neq a_n} \Delta t_k + 2M \sum (b_n - a_n).$$

$$\leq \varepsilon'(b-a) + 2M\varepsilon' \quad \text{due to } (*).$$

$$= \varepsilon'(b-a+2M).$$

Choose $\varepsilon' = \dfrac{\varepsilon}{b-a+2M}$

Then for P,

$$U(P,f) - L(P,f) \leq \varepsilon.$$

So for any $\varepsilon > 0$, we can construct P s.t above is true. $\Rightarrow f \in R[a,b]$.

- Properties of the Integral

Th^m: 1) If $f_1, f_2 \in R[a,b]$ then $c_1 f_1 + c_2 f_2 \in R[a,b]$ $\forall\, c_1, c_2 \in \mathbb{R}$

Moreover

$$\int_a^b (c_1 f_1 + c_2 f_2)\,dt = c_1 \int_a^b f_1(t)\,dt + c_2 \int_a^b f_2(t)\,dt.$$

2) $f_1 \le f_2$ on $[a,b] \Longrightarrow$

$$\int_a^b f_1(t)\,dt \le \int_a^b f_2(t)\,dt$$

3) $f \in R[a,b]$ & $a < c < b \Rightarrow f \in R[a,c]$ & $f \in R[c,b]$

and:

$$\int_a^b f(t)\,dt = \int_a^c f(t)\,dt + \int_c^b f(t)\,dt.$$

Pf: Let us prove (3) for instance. Sps $f \in R[a,b]$

Given $\varepsilon > 0$, $\exists$ partition P of $[a,b]$ s.t

$$U(P,f) - L(P,f) < \varepsilon$$

If P does not contain c, we can consider a refinement $P^* = P \cup \{c\}$. Since

$$U(P^*,f) < U(P,f)$$

$$L(P^*,f) > L(P,f)$$

we still have $U(P^*,f) - L(P^*,f) < \varepsilon$. $(*)$

Then we can write

$$P^* = P_1 \cup P_2$$

where
$$P_1 = \{t \in P^* \mid t \le c\}$$
$$P_2 = \{t \in P^* \mid t \ge c\}$$

So P_1, P_2 are partition of $[a,c]$ & $[c,b]$ resp. Clearly (since $P_1 \cap P_2$ is a single point & $t \le s \ \forall\ t \in P_1,\ s \in P_2$).

$$U(P^*, f) = U(P_1, f) + U(P_2, f).$$
$$L(P^*, f) = L(P_1, f) + L(P_2, f).$$

But $U(P_1, f) - L(P_1, f)$ & $U(P_2, f) - L(P_2, f) > 0$

$\Rightarrow$ So $(*) \Rightarrow$
$$U(P_1, f) - L(P_1, f) < \varepsilon$$
$$U(P_2, f) - L(P_2, f) < \varepsilon.$$

$\Rightarrow f \in R[a,c],\ R[c,b].$

Similarly it is easy to see
$$\int_a^b f(t)\,dt = \int_a^c f(t)\,dt + \int_c^b f(t)\,dt.$$

$\underline{\text{Th}^m}$ Sps $f \in R[a,b]$, $m \le f \le M$ & φ is cont. on $[m, M]$

Then $\varphi \circ f \in R[a,b]$.

$\underline{\text{Pf}}$ Fix $\varepsilon > 0$. Let $\varepsilon' > 0$ (which will depend on ε & we'll pick later) Denote $\varphi \circ f(*) = h(*)$.

Since φ is uniformly cont. on $[m, M]$. $\exists\ \delta < \varepsilon'$ s.t

$$|s - t| < \delta \implies |\varphi(s) - \varphi(t)| < \varepsilon' \qquad (*)$$

$f \in R[a,b] \implies \exists\ P = \{t_0, t_1, \dots, t_n\}$ s.t

$$U(P,f) - L(P,f) < \delta^2 \qquad (**)$$

$\underline{\text{Claim}}$: $U(P, h) - L(P, h) < \varepsilon$ if $\varepsilon' = \dfrac{\varepsilon}{b - a + 2K}$

where $M = \sup|\varphi(t)|$.

$\underline{\text{Pf}}$: Denote $M_k = \sup\limits_{[t_{k-1}, t_k]} f$, $m_k = \inf\limits_{[t_{k-1}, t_k]} f$

$$M_k^* = \sup_{[t_{k-1}, t_k]} f \qquad m_k^* = \inf_{[t_{k-1}, t_k]} f$$

$\underline{\text{Good Set / Bad Set}}$:

$$G = \{k \in \{1, \dots, n\} \mid M_k - m_k < \delta\}$$

$$B = \{k \in \{1, \dots, n\} \mid M_k - m_k \ge \delta\}$$

Recall $U(P, h) - L(P, h) = \sum (M_k^* - m_k^*) \Delta t_k$.

CASE 1 $k \in G$.

Sps $M_k^* = \varphi(f(\bar{t}_k))$, $m_k^* = \varphi(f(\underline{t}_k))$, $\bar{t}_k, \underline{t}_k \in [t_{k-1}, t_k]$

Since $M_k - m_k < \delta \Rightarrow |f(\bar{t}_k) - f(\underline{t}_k)| < \delta$.

$$(*) \Rightarrow \boxed{M_k^* - m_k^* < \varepsilon'} \Rightarrow \sum_{k\in G} (M_k^* - m_k^*) \Delta t_k \le \varepsilon'(b-a).$$

CASE 2: $k \in B$.

$$M_k^* - m_k^* \le 2M.$$

Subclaim: $\sum_{k\in B} \Delta t_k < \varepsilon'$.

Pf:
$$\delta \sum_{k\in B} \Delta t_k \le \sum_{k\in B} (M_k - m_k)\Delta t_k$$
$$\le \sum_k (M_k - m_k)\Delta t_k = U(P,f) - L(P,f)$$
$$< \delta^2$$
$$\Rightarrow \Delta t_k < \delta < \varepsilon'.$$

So now,
$$\sum_{k\in B} (M_k^* - m_k^*)\Delta t_k < 2M\varepsilon'$$
$$\Rightarrow U(P,h) - L(P,h) = \sum_{k\in G} (M_k^* - m_k^*)\Delta t_k + \sum_{k\in B} (M_k^* - m_k^*)\Delta t_k$$

$$\leq \varepsilon' [b - a + 2M]$$

$$< \varepsilon \quad \text{if} \quad \varepsilon' = \varepsilon / b - a + 2M$$

Done!

Rk: If φ is merely assumed to be integrable, it is no longer true that $\varphi \circ f$ is integrable.

Cor: 1) $f, g \in R[a,b] \Rightarrow fg \in R[a,b]$.

2) $f \in R[a,b] \Rightarrow |f| \in R[a,b]$ Moreover

(Δ-ineq) $$\left| \int_a^b f(t)\,dt \right| \leq \int_a^b |f|\,dt.$$

Pf: 1) $f \in R[a,b] \Rightarrow f^2 \in R[a,b]$. (Apply theorem to $\varphi(t) = t^2$). But then $f, g \in R[a,b] \Rightarrow (f+g)^2, (f-g)^2 \in R[a,b]$

$$fg = \frac{1}{2}\left[(f+g)^2 - (f-g)^2\right] \in R[a,b]$$

2) $f \in R[a,b] \Rightarrow |f| \in R[a,b]$ follows by applying Th^m to $\varphi(t) = |t|$.

Choose $c = \pm 1$ s.t $c \int_a^b f\,dt \geq 0$.

Then $$\left| \int_a^b f(t)\,dt \right| = c \int_a^b f(t)\,dt = \int_a^b (cf)\,dt \leq \int_a^b |f|\,dt$$

since $cf \leq |f|$.

• <u>Integration & Diff</u>

<u>Th^m (1st fundamental Th^m)</u> : Let $f \in R[a,b]$. For $a \le x \le b$, put

$$F(x) = \int_a^x f(t)\,dt$$

Then (1) F is cont on $[a,b]$.

(2) If f is cont at $p \in (a,b)$, then F is diff at p &

$$F'(p) = f(p)$$

<u>Pf</u>: 1) Sps $|f(t)| \le M \ \forall\ t \in [a,b]$. If $a \le x < y \le b$

$$|F(y) - F(x)| = \left| \int_x^y f(t)\,dt \right| \le \int_x^y |f(t)|\,dt \le M(y-x)$$

Given $\varepsilon > 0$, if $\delta = \varepsilon / M$

$|y - x| < \delta \implies |F(y) - F(x)| \le \varepsilon$.

So F is cont.

2) Sps f is cont at p. Given $\varepsilon > 0$, $\exists\ \delta$ s.t

$|t - p| < \delta \implies |f(t) - f(p)| < \varepsilon$.

Consider for $x > p$.

$$\frac{F(x) - F(p)}{x - p} - f(p) = \frac{1}{x-p} \int_p^x f(t)\,dt - \frac{f(p)}{(x-p)} \int_p^x dt$$

$$\Rightarrow \left| \frac{F(x) - F(p)}{x - p} - f(p) \right| \leq \frac{1}{(x-p)} \left| \int_p^x [f(t) - f(p)] \, dt \right|$$

If $|x-p| \leq \delta \Rightarrow |t-p| < \delta \quad \forall \quad t [p, x]$.

$$\Rightarrow \quad |f(t) - f(p)| < \varepsilon$$

$$\Rightarrow \quad \frac{1}{(x-p)} \left| \int_p^x f(t) - f(p) \, dt \right| \leq \frac{1}{(x-p)} \int_p^x |f(t) - f(p)| \, dt$$

$$< \frac{\varepsilon (x-p)}{x-p} = \varepsilon.$$

Similar arg. works if $p > x$.

$$\Rightarrow \lim_{x \to p} \frac{F(x) - F(p)}{x - p} = f(p) \Rightarrow F'(p) = f(p).$$

<u>Th^m</u> (2^nd fund. theorem) If $F : [a,b] \to \mathbb{R}$ cont. & diff on (a,b) s.t

$$f(x) = F'(x) \in R[a,b].$$

Then
$$\int_a^b f(x) \, dx = F(b) - F(a)$$

<u>Pf</u>: $f \in R[a,b] \Rightarrow \forall \varepsilon > 0, \exists$ partition $P = \{x_0, \dots, x_n\}$ s.t.

$$U(P,f) - L(P,f) := \sum_{k=1}^{n} (M_k - m_k) \Delta x_k < \frac{\varepsilon}{2}.$$

MVT $\Rightarrow \exists \; t_k \in [x_{k-1}, x_k]$ s.t

$$F(x_k) - F(x_{k-1}) = F'(t_k)\,\Delta x_k$$
$$= f(t_k)\,\Delta x_k$$

<u>Note</u>

$$F(b) - F(a) = F(b) - \cancel{F(x_{n-1})} + \cancel{F(x_{n-1})} - \cancel{F(x_{n-2})} + \cancel{\cdots} + \cancel{F(x_1)} - F(a)$$

$$= \sum_{k=1}^{n} f(t_k)\,\Delta x_k$$

<u>Claim</u>:

$$\left| \sum_{k=1}^{n} f(t_k)\,\Delta x_k - \int_a^b f(x)\,dx \right| < \varepsilon$$

<u>Pf</u>: Since $L(P,f) \le \int_a^b f(x)\,dx \le U(P,f)$ $\forall$ partition

$$\Rightarrow \left| U(P,f) - \int_a^b f(x)\,dx \le \right| < \varepsilon/2$$

Also, $L(P,f) \le \sum_{k=1}^{n} f(t_k)\,\Delta x_k \le U(P,f)$

$$\Rightarrow \left| U(P,f) - \sum_{k=1}^{n} f(t_k)\,\Delta x_k \right| < \frac{\varepsilon}{2}$$

Δ-ineq $\Rightarrow$

$$\left| \sum_{k=1}^{n} f(t_k)\,\Delta x_k - \int_a^b f(x)\,dx \right| < \varepsilon$$

So $\forall \ \varepsilon > 0$.

$$\left| F(b) - F(a) - \int_a^b f(x)\,dx \right| < \varepsilon$$

$$\Rightarrow \quad F(b) - F(a) = \int_a^b f(x)\,dx$$

• Indefinite integrals

Def^n: A function F s.t

$$F'(x) = f(x)$$

$\forall \ x \in [a,b]$ is called the indefinite integral of f on $[a,b]$.

Rk: Indefinite integrals are unique up to constants

If F & G are integrals of f, then

$$F'(x) = f(x) = G'(x) \Rightarrow (F-G)' \equiv 0$$

$$\Rightarrow \quad F - G = \text{const}.$$

In view of the remark, we write, somewhat ambiguously

$$F(x) = \int f(x)\,dx + C.$$

$$\& \quad \int_a^b f(x)\,dx = F(b) - F(a).$$

Example 1) $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ (since $\frac{d}{dx}\frac{x^{n+1}}{n+1} = x^n$), $n \neq -1$

2) $\int \sin x\,dx = -\cos x + C$.

$\int \cos x\,dx = \sin x + C$.

3) $\int e^x\,dx = e^x + C$

4) $\int \frac{dx}{x} = \ln|x| + C$ on $\mathbb{R}\setminus\{0\}$

- Integration by parts

Thm: Sps F, G diff on (a,b) & cont on $[a,b]$ s.t. $f = F' \in R[a,b]$ & $g = G' \in R[a,b]$. Then

$$\int_a^b F(t)\,g(t)\,dt = F(b)G(b) - F(a)G(a) - \int_a^b G(t)\,f(t)\,dt.$$

Pf: Let $H(x) = \int_a^x F(t)g(t)\,dt + \int_a^x G(t)f(t)\,dt - F(x)G(x) + F(a)G(a)$.

$$H'(x) = \cancel{F(x)g(x)} + \cancel{G(x)f(x)} - \cancel{F(x)G'(x)} - \cancel{F'(x)G(x)}$$

$= 0$

$\Rightarrow H(x) = \text{const} \Rightarrow H(b) = H(a)$

But $H(a) = 0 \Rightarrow H(b) = 0$

But $H(b) = \int_a^b F(t)\, g(t)\, dt + \int_a^b G(t)\, f(t)\, dt - F(b)\, G(b) + F(a) G(a)$

$= 0$

Done!

Rk ① Easier to remember as diff forms. Set

$F = u, \quad G = v$,

$$g(x)\,dx = G'(x)\,dx = dG(x) = dv$$

$$f(x)\,dx := du.$$

Then the theorem, at the level of indefinite integrals, says

$$\boxed{\int u\,dv = uv - \int v\,du}$$

② Integration by parts is like the "product" rule for integration.

Example: Compute $\int \ln x \, dx$

$$\int \ln x\, dx = \int \ln x \cdot dx$$

Put $u = \ln x \quad dv = dx$

$du = \frac{dx}{x}, \quad v = x$.

$$= x \ln x - \int \cancel{x} \cdot \frac{dx}{\cancel{x}}$$

$$= x \ln x - \int dx$$

$$= x \ln x - x + C.$$

Change of Variables

Th^m If $u = g(t) : [c, d] \longrightarrow [a, b]$ s.t. g' is cont & $g' \neq 0$
If f is cont. on $[a, b]$, then for any $x \in [c, d]$

$$\int_{g(c)}^{g(x)} f(u)\, du = \int_c^x f(g(t)) \cdot g'(t)\, dt$$

Pf: Let

$$F(v) = \int_{g(c)}^{v} f(u)\, du, \quad G(x) = \int_c^x f(g(t)) \cdot g'(t)\, dt$$

Claim: $G(x) = F(g(x))$.

Pf: $G'(x) = f(g(x))\, g'(x)$, $F'(v) = f(v)$.

$$\frac{d}{dx} F(g(x)) = F'(g(x)) \cdot g'(x)$$

$$= f(g(x)) \cdot g'(x).$$

So $[G(x) - F(g(x))]' = 0 \Rightarrow G(x) = F(g(x)) + A$.

But at $x = c$, $F(g(c)) = 0$, $G(c) = 0 \Rightarrow A = 0$

This proves the claim & Th^m.

Example: $\displaystyle\int_0^{\pi/4} \tan t \, dt = \int_0^{\pi/4} \frac{\sin t}{\cos t} \, dt$

Let $u = \cos t, \Rightarrow du = -\sin t \, dt$, $\quad t = 0,\ u = 1$; $\quad t = \pi/4,\ u = 1/\sqrt{2}$.

$$\int_0^{\pi/4} \frac{\sin t}{\cos t} \, dt = \int_1^{1/\sqrt{2}} -\frac{du}{u} = \left| \frac{1}{u^2} \right|_1^{1/\sqrt{2}} = 2 - 1 = 1.$$

• Improper Integrals

Def^n: 1) Let $f \in R[a,R]$ $\forall$ $a \le R < b$, where $b \in (a, \infty]$.

Then we define

$$\int_a^b f(t) \, dt = \lim_{R \to b^-} \int_a^R f(t) \, dt$$

if the limit exists & is finite. We then say $f \in R[a,b]$.

2) If $f \in R[r, b]$ $\forall$ $a < r \le b$ where $a \in [-\infty, b)$.

we define

$$\int_a^b f(t) dt = \lim_{r \to a^+} \int_r^b f(t) dt$$

if limit exists & is finite

Example: 1) $\displaystyle\int_0^1 \frac{dt}{\sqrt{t}} = \lim_{r \to 0^+} \int_r^1 \frac{dt}{\sqrt{t}} = \lim_{r \to 0^+} 2\sqrt{t} \Big|_r^1$

$$= 2 - 2 \lim_{r \to 0^+} \sqrt{r} = 2.$$

2) More generally

$$\int_0^1 x^p\,dx = \begin{cases} \dfrac{1}{p+1} & p > -1 \\ \text{DNE} & p \le -1 \end{cases}$$

$$\int_1^\infty x^p\,dx = \begin{cases} -\dfrac{1}{p+1} & p < -1 \\ \text{DNE} & p \ge -1 \end{cases}$$

Th^m (Comparison principle). If $f(x) \ge g(x) \ge 0$ for $x \in [a, b)$ and f, g cont. on $[a, b)$.

1) $\int_a^b f$ convergent $\Rightarrow$ $\int_a^b g$ conv.

2) $\int_a^b g$ divergent $\Rightarrow$ $\int_a^b f$ divergent

· Application to infinite series

Th^m: Let $f : [1, \infty] \to [0, \infty)$ be a decreasing function s.t. $\lim_{x \to \infty} f(x) = 0$. For $n = 1, 2, \ldots$ define

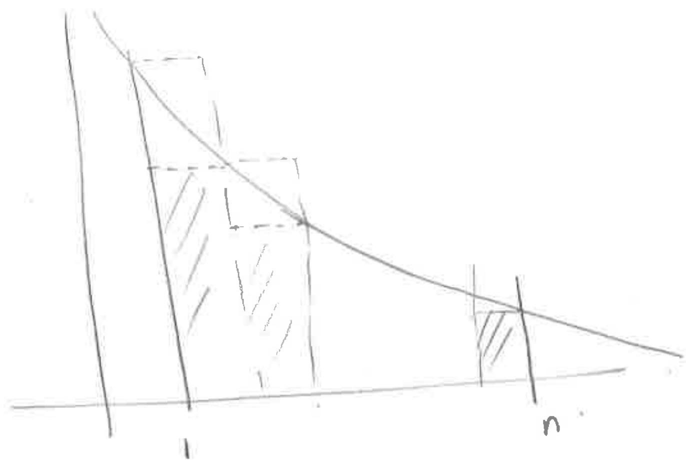
$$S_n = \sum_{k=1}^{n} f(k), \quad I_n = \int_1^n f(x)\,dx$$

Then

(1) $f(n) + I_n \leq S_n \leq f(1) + I_n$.

(2) $\sum_{k=1}^{n} f(k)$ conv. $\Longleftrightarrow$ $\int_1^{\infty} f(x)\,dx$ conv.

Pf:



(1) $I_n = \int_1^n f(t)\,dt = \sum_{k=1}^{n-1} \int_k^{k+1} f(t)\,dt$

On $[k, k+1]$, $f(t) \leq f(k)$.

$\Rightarrow \int_k^{k+1} f(t)\,dt \leq f(k)$

$\Rightarrow I_n \leq \sum_{k=1}^{n-1} f(k) = S_{n-1} = S_n - f(n)$.

$\Rightarrow I_n + f(n) \leq S_n$.

On the other hand

$I_n = \int_1^n f(t)\,dt = \sum_{k=1}^{n-1} \int_k^{k+1} f(t)\,dt \geq \sum_{k=1}^{n-1} f(k+1)$.

$$= \sum_{j=2}^{n} f(j) = S_n - f(1)$$

$$\Rightarrow \qquad S_n \leq I_n + f(1).$$

This proves (I).

(II). Since $f(x) \geq 0$, $\sum_{k=1}^{\infty} f(k)$ conv. $\Leftrightarrow$ $\{S_n\}$ is bounded

$\int_1^{\infty} f(t)dt$ conv $\Leftrightarrow$ $\{I_n\}$ is bounded.

But $\{S_n\}$ is bdd. $\Leftrightarrow$ I_n is bounded since

$f(n) \to 0$ as $n \to \infty$.

Done!

<u>Example</u>. $\sum \frac{1}{n^p}$ conv. $\Leftrightarrow$ $p > 1$.

Consider $f(x) = x^{-p}$. Then $f(x) \geq 0$ & $f \searrow 0$.

as $x \to +\infty$.

Applying Th^m. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ conv. $\Leftrightarrow$ $\int_1^{\infty} \frac{dx}{x^p}$ conv.

By earlier example, this happens $\Leftrightarrow$ $-p < -1$

or $p > 1$.