INFINITE SERIES - 2

• Dirichlet's test

Th^m Suppose

(a) Partial sums A_n of $\sum a_n$ are bounded.

(b) $b_n \downarrow 0$.

Then $\sum a_n b_n$ converges

Lemma (Summation by parts). Given $\{a_n\}$, $\{b_n\}$, put

$$A_n = \sum_{k=1}^{n} a_k$$

Then for $0 < p \le q$

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p$$

Pf: Note that $a_n = A_n - A_{n-1}$, so

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q} (A_n - A_{n-1}) b_n$$

$$= \sum_{n=p}^{q} A_n b_n - \sum_{n=p}^{q} A_{n-1} b_n$$

$$= \sum_{n=p}^{q-1} A_n b_n - \sum_{m=p-1}^{q-1} A_m b_{m+1} + A_q b_q$$

$$= \sum_{n=p}^{q-1} A_n b_n - \sum_{n=p}^{q-1} A_n b_{n+1} + A_q b_q - A_{p-1} b_p$$

Pf of Th^m: Choose M s.t $|A_n| \le M$ for all n. Fix $\varepsilon > 0$.

$$\left|\sum_{n=p}^{q} a_n b_n\right| = \left|\sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p\right|$$

Sps $N < p \le q$. Then ~~since~~ $\{b_n\}$ decreasing, $b_n - b_{n+1} \geq 0$

So,

$$\left|\sum_{n=p}^{q} a_n b_n\right| \le M\left[\sum_{n=p}^{q-1} (b_n - b_{n+1}) + b_q + b_p\right]$$

$$= M\left[b_p - b_q + b_q + b_p\right]$$

$$= 2Mb_p \le 2Mb_N.$$

Since $b_n \downarrow 0$, choose N s.t $b_N \le \varepsilon/2M$

Then for $p, q \geq N$,

$$\left|\sum_{n=p}^{q} a_n b_n\right| \le \varepsilon$$

Cauchy criteria $\Longrightarrow$ $\sum a_n b_n$ converges.

Corollary: If $b_n \downarrow 0$, then

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n$$

converges.

Def^n: Such a series is called an alternating series

If $\theta \neq 0$ (i.e $z \neq 1$)

$$\sum_{k=1}^{n} z^k = \sum_{k=1}^{n} e^{ik\theta} = \frac{\sin(n\theta/2)}{\sin(\theta/2)} \cdot e^{i(n+1)\theta/2}$$

Clearly the right hand side has no limit

So $\sum z^n$ diverges for all z s.t $|z|=1$.

- Rearrangements. Consider the series

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

and a rearrangement

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{16} \cdots$$

Turns out (assignment problem) that the rearrangement also converges. Denote it's sum and partial sums by S', S'_n resp

Now,

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \left(\frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \cdots\right)$$

$$\leq 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6} \Longrightarrow S \leq 5/6$$

Next, we can write

$$S = \left(1 + \frac{1}{3} - \frac{1}{2}\right) + \left(\frac{1}{5} + \frac{1}{7} - \frac{1}{4}\right) + \cdots$$

So
$$S'_{3n} = \sum_{k=1}^{n} \left(\frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k} \right).$$

Since $\frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k} > 0$.

Clearly $S'_3 < S'_6 < S'_9 < \cdots$

$$\Rightarrow \quad S' = \limsup_{n \to \infty} S'_n > S'_3 = 1 + \frac{1}{3} - \frac{1}{2} = \frac{5}{6}.$$

So $S' > 5/6$.

$S < 5/6$.

$$\Rightarrow \quad \boxed{S \neq S'}$$

The reason this happens, is that $\sum (-1)^{k+1}/k$ is <u>conditionally convergent</u>.

Def^n: Let $\sum a_n$ be an infinite series, and

$$\sigma : \mathbb{N}\setminus\{0\} \longrightarrow \mathbb{N}\setminus\{0\}.$$

a bijection. If we denote

$$a_n' = a_{\sigma(n)},$$

the series $\sum a_n'$ is called a re-arrangement of $\sum a_n$.

Example: Let $a_n = \dfrac{(-1)^{n+1}}{n}$, $\sum a_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6}$

$$\sigma(n) = \begin{cases} 4\left(\frac{n-1}{3}\right)+1 & 3 \mid n-1. \\ 4\left(\frac{n-2}{3}\right)+3. & 3 \mid n-2. \\ 2n/3 & 3 \mid n. \end{cases}$$

Check: This is a bijection.

Then $\sigma(1) = 1$, $\sigma(2) = 3$, $\sigma(3) = 2$.
$\sigma(4) = 5$ $\sigma(5) = 7$, $\sigma(6) = 4$.

So rearranged series is

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots$$

So a rearrangement is simply a re-ordering or a permutation of the series

$\underline{\text{Th}^m}$: Let $\sum a_n$ be a convergent series.

(1) If $s = \sum a_n$, and $\sum a_n$ is <u>absolutely convergent</u>

Then any re-arrangement $\sum a_n'$ also converges and infact

$$\sum a_n' = s.$$

(2) If $\sum a_n$ is <u>conditionally convergent</u>, then for any $L \in [-\infty, \infty]$, there is a re-arrangement $\sum a_n'$ s.t

$$\sum a_n' = L.$$

<u>Pf</u>: (1) $\sum a_n$ is <u>abs. convergent</u>. Let $\sum a_n'$ be any re-arrangement

<u>Step 1</u>: $\sum a_n'$ converges.

<u>Pf</u>: Let $\sigma : \mathbb{N}\setminus\{0\} \to \mathbb{N}\setminus\{0\}$ s.t $a_n' = a_{\sigma(n)}$.

Then $\displaystyle\sum_{k=1}^{n} |a_k'| < \sum_{k=1}^{\infty} |a_k| < \infty$ since $\sum |a_k|$ is conv.

So $\displaystyle\sum_{k=1}^{\infty} |a_k'|$ is a series with bounded partial sums and positive terms $\Rightarrow \sum |a_k'|$ converges $\Rightarrow \sum a_k'$ converges.

Step 2: $\sum a_n' = S$.

Pf: Let $t_n = \sum_{k=1}^{n} a_k'$, $S_m = \sum_{k=1}^{m} a_k$.

Aim: For any $\varepsilon > 0$, find N s.t $\forall n > N$,

$$|S - t_n| < \varepsilon.$$

Given $\varepsilon > 0$, $\exists$ M s.t $\forall m > M$,

(*). $$\sum_{k=m}^{\infty} |a_k| < \frac{\varepsilon}{2}$$ (since $\sum a_n$ is abs convergent).

In particular
$$|S - S_m| = \left|\sum_{k=m}^{\infty} a_k\right| \leq \sum_{k=m}^{\infty} |a_k| \leq \frac{\varepsilon}{2}.$$

Recall $a_n' = a_{\sigma(n)}$. Choose N big enough so that

$\{1, 2, 3, \ldots, M\} \subseteq \{\sigma(1), \ldots, \sigma(N)\}$.

So if $n > N$ $\Rightarrow \{1, 2, \ldots, M\} \subseteq \{\sigma(1), \ldots, \sigma(n)\}$.

For $n > N$,

$$|t_n - S_M| = |a_{\sigma(1)} + \cdots + a_{\sigma(n)} - a_1 - a_2 - \cdots - a_M|$$

$$\leq \sum_{k=M+1}^{\infty} |a_k| < \frac{\varepsilon}{2} \quad \text{(by *)}.$$

Then
$$|t_n - S| \leq |t_n - S_M| + |S_M - S| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\ ; \forall n > N$$

Done!

(b) (Idea of proof). Define

$$p_n = \frac{|a_n| + a_n}{2}, \quad q_n = \frac{|a_n| - a_n}{2}.$$

So $p_n = \begin{cases} a_n, & a_n \geq 0 \\ 0 & a_n < 0 \end{cases}$, $q_n = \begin{cases} -a_n & a_n \leq 0 \\ 0 & a_n \geq 0. \end{cases}$

Step 1 $\sum p_n$ and $\sum q_n$ diverge.

Pf: Sps both converge. Then since $|a_n| = (p_n + q_n)$ we would have that $\sum |a_n|$ converges, a contradiction since $\sum a_n$ is conditionally convergent.

Sps only one of them, say $\sum p_n$ converges. Since $a_n = p_n - q_n$, for any N,

$$\sum_{n=1}^{N} a_n = \sum_{n=1}^{N} p_n - \sum_{n=1}^{N} q_n.$$

Since $\sum a_n$ converges, $\sum p_n$ converges, $\sum q_n$ will be forced to converge, and we are back to the earlier case.

Step 2: For simplicity, sps $L > 0$.

Since $\sum p_n$ diverges, $\exists N$ s.t $\forall n > N$

$$\sum_{k=1}^{n} p_k > L.$$

Let $N_1 =$ min. # s.t $\sum_{k=1}^{N_1} p_k > L$.

i.e $$\sum_{k=1}^{N_1-1} p_k \leq L \quad \text{but} \quad \sum_{k=1}^{N_1} p_k > L.$$

Next we add enough −ve terms s.t the sum is just less than L. i.e choose M_1 s.t

$$\sum_{k=1}^{N_1} p_k - \sum_{k=1}^{M_1} q_k < L.$$

but $$\sum_{k=1}^{N_1} p_k - \sum_{k=1}^{M_1-1} q_k \geq L.$$

Let N_2, M_2 be smallest integers s.t

$$p_1 + \cdots + p_{N_1} - q_1 - \cdots - q_{M_1} + p_{N_1+1} + \cdots + p_{N_2} > L.$$

$$p_1 + \cdots + p_{N_1} - q_1 \cdots - q_{M_1} + p_{N_1+1} + \cdots + p_{N_2} - q_{M_1+1} - q_{M_1+2} - \cdots - q_{M_2} < L.$$

If x_k is the partial sum whose final term is p_{N_k} Then $|x_k - L| \leq p_{N_k} \xrightarrow{k\to\infty} 0$ since $\sum a_n$ conv.

If y_k is the partial sum whose final term is q_{M_k}, then $|y_k - L| \leq q_{M_k} \xrightarrow{k\to\infty} 0$.

If S_n's are the partial sums of the re-arrangem

If $N_k + 1 \leq n < N_{k+1}$.

Then $|S_n' - L| < q_{M_k}$.

If $M_k + 1 \leq n < M_{k+1}$.

$|S_n' - L| < p_{N_k}$.

So in any case $S_n' \to L$. Done!

<u>Example</u>: <u>A divergent rearrangement</u>: Consider a rearrangement

$$\sum a_n' = 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{5} - \frac{1}{4} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6}$$

$$+ \frac{1}{13} + \frac{1}{15} + \frac{1}{17} + \frac{1}{19} - \frac{1}{8} + \cdots$$

First an odd term is followed by even

Then <u>2</u> odd terms with <u>one</u> even

Then <u>3</u> odd terms with <u>one</u> even.

In general $\sum a_n' = \sum_{k=1}^{\infty} \left(\frac{1}{k(k-1)+1} + \cdots + \frac{1}{k(k-1)+2k-2} - \frac{1}{2k} \right)$.